Renormalisation in Perturbative Quantum Gravity

DISSERTATION

zur Erlangung des akademischen Grades

doctor rerum naturalium
(Dr. rer. nat.)
im Fach Physik

eingereicht an der
Mathematisch-Naturwissenschaftlichen Fakultät I
der Humboldt-Universität zu Berlin

von
Dipl.-Phys. Andreas Rodigast

Präsident der Humboldt-Universität zu Berlin:
Prof. Dr. Jan-Hendrik Olbertz

Dekan der Mathematisch-Naturwissenschaftlichen Fakultät I:
Prof. Dr. Andreas Herrmann

Gutachter:
1. Prof. Dr. Jan Plefka
2. Prof. Dr. Dietmar Ebert
3. Prof. Dr. Jan Pawlowski

Tag der mündlichen Prüfung: 2. Juli 2012

Abstract

In this thesis, we derive the gravitational one-loop corrections to the propagators and interactions of the Standard Model field. We consider a higher dimensional brane world scenario: Here, gravitons can propagate in the whole D dimensional space-time whereas the matter fields are confined to a d dimensional sub-manifold (brane).

In order to determine the divergent part of the one-loop diagrams, we develop a new regularisation scheme which is both sensitive for polynomial divergences and respects the Ward identities of the Yang-Mills theory.

We calculate the gravitational contributions to the β functions of non-Abelian gauge theories, the quartic scalar self-interaction and the Yukawa coupling between scalars and fermions. In the physically interesting case of a four dimensional matter brane, the gravitational contributions to the running of the Yang-Mills coupling constant vanish. The leading contributions to the other two couplings are positive. These results do not depend on the number of extra dimensions.

We further compute the gravitationally induced one-loop counterterms with higher covariant derivatives for scalars, Dirac fermions and gauge bosons. In is shown that these counterterms do not coincide with the higher derivative terms in the Lee-Wick standard model. A possible connection between quantum gravity and the latter cannot be inferred.

Zusammenfassung

In dieser Arbeit berechnen wir die gravitativen Ein-Schleifen-Korrekturen zu den Propagatoren und Wechselwirkungen der Felder des Standardmodells der Elementarteilchenphysik. Wir betrachten hierzu ein höherdimensionales *brane world*-Modell: Wärend die Gravitonen, die Austauchteilchen der Gravitationswechselwirkung, in der gesamten D-dimensionalen Raumzeit propagieren können, sind die Materiefelder an eine d-dimensionale Untermanigfaltigkeit (*brane*) gebunden.

Um die divergenten Anteile der Ein-Schleifen-Diagramme zu bestimmen, entwickeln wir ein neues Regularisierungschema welches einerseits die Wardidentitäten der Yang-Mills-Theorie respektiert anderseits sensitiv für potenzartige Divergenzen ist.

Wir berechnen die gravitativen Beiträge zu den β-Funktionen der Yang-Mills-Eichtheorie, der quartischen Selbst-Wechselwirkung skalarer Felder und der Yukawa-Wechselwirkung zwischen Skalaren und Fermionen. Im physikalisch besonders interessanten Fall einer vier-dimensionalen Materie-*brane* verschwinden die gravitativen Beiträge zum Laufen der Yang-Mills-Kopplungskonstante. Die führenden Beiträge zum Laufen der anderen beiden Kopplungskonstanten sind positiv. Diese Ergebnisse sind unabhängig von der Anzahl der Extradimensionen in denen die Gravitonen propagieren können.

Des Weiteren bestimmen wir alle gravitationsinduzierten Ein-Schleifen-Konterterme mit höheren kovarianten Ableitungen für skalare Felder, Dirac-Fermionen und Eichbosonen. Ein Vergleich dieser Konterterme mit den höheren Ableitungsoperatoren des Lee-Wick-Standardmodells zeigt, dass die Gravitationskorrekturen nicht auf letzte beschränkt sind. Eine Beziehung zwischen Quantengravitation und dem Lee-Wick-Standardmodell besteht somit nicht.

Contents

1. Introduction

Einstein's general relativity is one of the most beautiful physical theories. It starts from the very intuitive and simple Strong Equivalence Principle: A free falling observer cannot determine his velocity and position in space froqm local observables. A mathematical formulation of the principle is that a physical theory should be invariant under local diffeomorphism transformations. And although the principle does not say anything about attraction of masses, General Relativity yields an elegant and successful description of gravity on macroscopic scales. The other observed fundamental forces are described as gauge theories in a quite similar manner. Here, one does not demand the invariance under coordinate transformations (diffeomorphisms), but the invariance under internal symmetry transformations. This seemingly small difference has huge consequences when we quantise the field theories arising from these principles. The quantum gauge theories are renormalisable; General Relativity on the other hand is—in its perturbatively quantised form—non-renormalisable [1].

In a perturbative quantum field theory UV divergences can be cancelled by adding counterterms to the action. The counterterms of a renormalisable theory have the same structure as the terms of the bare action. By the renormalisation of the parameters of the theory, the counterterms can be absorbed at each order of perturbation theory. In a non-renormalisable theory, the counterterms have an infinite number of different structures. Consequently, the renormalised action has infinitely many parameters.

Due to its non-renormalisability, perturbative quantum gravity is ill-suited as a fundamental theory at arbitrarily high energies. The coupling of the Einstein-Hilbert theory to any type of matter fields leads to perturbatively non-renormalisable theories as well [2–6]. Nevertheless, treated as an effective field theory, non-renormalisable theories can still be used for perturbative quantum field theory calculations. An effective field theory is characterised by an intrinsic UV cut-off scale and yields a valid description only for physical processes below this scale.

The effective field theory description of gravity was established by Donoghue [7, 8]. Therein perturbatively quantised Einstein gravity can be used to determine genuine predictions of quantum gravity for energies well below the Planck scale $M_{\text{Planck}} \approx 10^{19}\text{GeV}$. Hence, the effective field theory approach can provide both phenomenologically and methodologically interesting insight into the underlying quantum theory of gravitation, for a review see e. g., [9, 10].

In this context, Robinson and Wilczek [11] initiated an intriguing discussion on gravitational corrections to the running of gauge couplings calculated in the framework of effective field theories. They claimed to find gravitational corrections to the running of Abelian and non-Abelian gauge couplings, which would render all gauge theories, including QED, asymptotically free. However, in a careful reconsideration of the calculations Pietrykowski [12] proved that the background field method they used yields gauge dependent results. Many other work criticized and expanded the results of Robinson and Wilczek. At this point we give only a short overview of the literature. An extended discussion will follow in chapter 8. Toms [13] used the Vilkovisky-DeWitt background field method in Landau-

DeWitt gauge together with dimensional regularisation and concluded that the gravitational corrections to gauge couplings vanish. Later, Toms included a finite cosmological constant in a similar calculation [14] which yield a non-vanishing logarithmic contribution of the running of gauge couplings. The author and collaborators [15, 16] used Feynman graph techniques and compared dimensional and momentum cut-off regularisation. We found that both regularisation schemes yield a zero result. Later, we expanded our computations in a scenario with large extra dimensions [17]. These calculation are part of this thesis. Tang and Wu approached the problem using the rather obscure loop-regularisation in several works[18–20]. The first to use non-perturbative functional renormalisation group methods to calculate the running of the gauge coupling constant in the Einstein-Yang-Mills system were Daum, Harst and Reuter [21]. Several investigations by different authors in the Vilkovisky-DeWitt background field method followed [22–24], all coming to varying results. The regulator and gauge fixing dependence of the results were investigated by Folkerts, Litim and Pawlowski [25] using the functional renormalisation group. The possible influence of surface terms form the momentum cut-off was pointed out by Felipe *et al.* [26]. The physical interpretation of a possible quadratic energy dependence of running couplings was discussed in [27–30].

The discussion of the gravitational corrections to Maxwell and Yang-Mills gauge couplings is still vivid. The gravitational contributions to the running of the other renormalisable couplings of the Standard Model, namely, quartic scalar self-interaction and Yukawa coupling, on the other hand, attracted less interest. The first to approach this question were Zanusso, Zambelli, Vacca and Percacci [31] using functional renormalisation group methods. Independently, the author and collaborator [32] preformed a perturbative calculation using dimensional regularisation. In this thesis we extend these results using a momentum cut-off regularisation which is sensitive to power like divergences.

Hierarchy problem

The Standard Model of particle physics contains only renormalisable couplings, namely three gauge couplings, Yukawa and quartic scalar self interaction. Still, it cannot be a fundamental theory which is valid at arbitrarily high energy. It has to be modified already at scales which are well below the Planck scale where the effects of quantum gravity are not relevant. The reason is the following: The Standard Model's only explicit mass scale, the mass of the Higgs boson receives quadratic quantum corrections. Unless all Standard Model parameters are extremely fine tuned, the renormalisation of the mass will push it up to the fundamental scale of the theory. This is called the hierarchy problem.

The best solution is to introduce new physics at a scale slightly above the electroweak scale $M_{\text{weak}} \sim 1\text{TeV}$. The most prominent candidate is the Minimally Supersymmetric Standard Model (MSSM) and other super symmetric extensions.

In this work, we deal with two different possibilities: Large extra dimensions scenarios and the Lee-Wick Standard Model.

Large Extra Dimensions

The idea of a universe with more then the visible four space-time dimensions is motivated by superstring theories [33, 34]. To formulate a consistent quantum theory of fundamental superstrings these have to propagate in ten space-time dimensions. The most common approach is to compactify the additional space dimensions, i. e., these have small spatial expansion below our experimental resolution.

Superstring theories describe the Standard Model fields as open strings which might be confined to a four dimensional sub-manifold (3-brane). Gravitons on the other hand are described as closed strings which can freely move in the bulk, i. e., the whole space-time. If the Standard Model particles are confined to a 3-brane, the compactification radius R is only constrained by direct test of gravitational interactions. Hence, it might be of the order of 10^{-4}mm, i. e., enormously large in comparison with the length scales of particle physics.

The interesting feature of this set-up is that large distance gravity is significantly weakened on the 3-brane. One can see this easily in a classical, non-relativistic example: Let us consider two test particles with masses m_1 and m_2 at distance r in D dimensions. From Gauss's law, we know that the gravitation potential between them is

$$V(r) = \frac{1}{8\pi M_{\mathrm{Planck}\,(D)}^{D-2}} \frac{m_1 m_2}{r^{D-3}} \,.$$

Now, we compactify $D - 4$ space dimension with compactification radius R. At small distances $r \ll R$, the potential is not altered. If the particles are placed at a distance $r \gg R$ the gravitational flux cannot penetrate the compactified dimension farther then the radius R and the potential is modified and becomes

$$V(r) = \frac{1}{8\pi M_{\mathrm{Planck}\,(D)}^{D-2} (2\pi R)^{D-4}} \frac{m_1 m_2}{r} \,,$$

which is the usual Coulomb potential in four dimensional space-time with a reduced coupling constant. This relation leads to the definition of an effective Planck mass on the 3-brane:

$$M_{\mathrm{Planck}\,(4)} = M_{\mathrm{Planck}\,(D)}^{D-2} (2\pi R)^{D-4}. \tag{1.1}$$

In such an extra dimensional scenario, the fundamental scale of quantum gravity is not the usual Planck mass $M_{\mathrm{Planck}\,(4)} \approx 10^{19}$GeV we know from our experience in the four-dimensional space-time, but $M_{\mathrm{Planck}\,(D)}$ which describes the gravitational interactions in D dimensions. From equation (1.1) we see that this scale might be as low as some TeV without contradicting current experimental data. Since the fundamental gravitational scale $M_{\mathrm{Planck}\,(D)}$ is an upper bound for all perturbative quantum field theories, extra dimensional set-ups are candidates for the new physics which can solve the hierarchy problem.

The Standard Model is unaltered by the presence of the extra dimensions, as long we consider energies below $M_{\mathrm{Planck}\,(D)}$. However, one might just as well consider the existence of extra "universal" compact dimensions for the brane fields. This was proposed for the first time in [35]. This universal extra dimensions scenario in the absence of gravity was considered by the authors of [36] who showed that the presence of extra dimensions for the Minimally Supersymmetric Standard Model fields leads (with a suitable cut-off procedure for the Kaluza-Klein towers of states) to a power law running of the MSSM couplings and grand unification at scales $M \ll 10^{16}$GeV well below the standard unification scale. A natural question to be addressed in this thesis is then how the running of gauge couplings is affected once one includes quantum gravitational effects in such a brane-world scenario.

The Lee-Wick Standard Model

One of the aspects of the non-renormalisable nature of perturbatively quantised gravity is the necessity to include higher derivative terms in the Lagrangian of the effective field theory. These higher derivative terms are interesting for themselves. A combination of higher derivative terms can provide a possible solution of the Standard Model hierarchy puzzle which was suggested by Grinstein, O'Connell, and Wise [37]. Their proposition is based on the ideas of Lee and Wick [38, 39], who studied the consequences of the assumption that the modification of the photon propagator in the Pauli-Villars regularisation [40] of quantum electrodynamics, corresponds to a physical degree of freedom. The modification of the photon propagator and thus the additional massive vector field correspond to a higher derivative term being added to the Lagrangian. Exploiting the improved UV behaviour Lee and Wick were able to construct a finite theory of quantum electrodynamics. Grinstein et al. extended the standard model to include special dimension-six higher derivative terms for each particle. These so called Lee-Wick terms have the special property of allowing for an equivalent formulation of the theory containing additional massive fields but only operators of dimension four or less. This property is crucial for the higher derivative theory fulfilling the constraints of perturbative unitarity [41]. The new particles are ghosts because their kinetic terms have the wrong sign. This indicates an instability on the classical level and results in problems with the unitarity for the quantum theory. However, these problems appear to be solvable and have been extensively discussed in the literature, for example in [38, 39, 42, 43].

The Lee-Wick terms used by Grinstein et al. are given by

$$\begin{aligned}
&\tfrac{1}{M_A^2}\,\mathrm{tr}\{(D^\mu F_{\mu\nu})^2\} && \text{for gauge fields,}\\
&\tfrac{1}{M_\phi^2}(D^2\phi)^\dagger(D^2\phi) && \text{for scalars (Higgs), and}\\
&\tfrac{i}{M_\psi^2}\overline{\psi}\,\not{D}^3\psi && \text{for fermions.}
\end{aligned}$$

This extension, known as the Lee-Wick standard model, is free of quadratic divergences and is therefore one possible solution to the hierarchy puzzle.

Shortly after its proposition Wu and Zhong [44] pointed out a possible connection between the Lee-Wick standard model and one-loop counterterms in the non-renormalisable Einstein-Maxwell theory.

Later, the same authors [45] claimed that a large extra dimension model provides a mechanism for the emergence of Lee-Wick partners, with masses in the TeV scale, for all particles. They base their arguments on the higher derivative counterterms that appear in the one-loop renormalisation of this theory, which according to their results are given by the Lee-Wick terms. However, they only calculated two-point functions which alone do not determine the higher derivative counterterms. In order to answer the question, whether the higher derivative counterterm correspond to the Lee-Wick terms or not, we calculate the gravitational one-loop corrections to the interactions of particles with gauge bosons.

Outline of this thesis

In chapter 2 we describe the effective field theory used for our calculations. We introduce the effective field theory set-up of quantum gravity including the large extra dimensions scenario. We note on the influence of fluctuations in the position of the brane and give a short sketch of the matter Lagrangians and how we obtain the Feynman rules for the coupling between the matter fields and the graviton. Finally, we discuss the energy expansion of the effective field theory and determine the range of energy and parameters (masses) in which our calculations are valid.

We continue in chapter 3 with a description of the Lee-Wick extension of the Standard Model and we derive a basis for the higher derivative operators and give the corresponding Feynman rules.

In chapter 4, we present the regularisation methods we developed for the computations of this thesis. In order to preserve gauge invariance, we develop a pre-regularisation scheme which allows us to maintain the gauge invariance of the one-loop amplitudes. We further show how the introduction of counterterms will lead to scale dependent coupling constants and mass terms and define the β functions which encode the running of the couplings. The last section of this chapter contains a small example of the implementation of the calculations in the computer algebra system FORM [46].

In chapter 5, we calculate the gravitational corrections to the wavefunction and mass renormalisation of the Standard Model fields, i. e., scalars, fermions and gauge bosons. In addition, we compute the renormalisation of the gauge covariant higher derivative terms introduced in chapter 3.

We determine the gravitationally induced running of the Standard Model couplings, i. e., gauge coupling, quartic scalar coupling and Yukawa coupling, in chapter 6.

Instead of continuing the perturbation expansion, we decided to approach the question using a functional renormalisation method, namely by calculating the flow equation of the effective average action. In chapter 7, We give a short introduction to the formalism of functional renormalisation and the background field formalism for gauge theories with a special focus on the Einstein-Yang-Mills system. Finally, we present some of the FORM routines we developed for background field calculations.

The various results and critism on the gravitational corrections to running of gauge couplings in the literature is disscussed in chapter 8.

2. Effective Field Theory of Gravity

In this chapter we describe the effective field theory used for our calculations. First, we introduce the effective field theory set-up of quantum gravity including the large extra dimensions scenario. We consider a scenario of a flat D dimensional space-time with d non-compact space-time directions and δ space-like dimensional compactified on a torus. The gravitational fields propagate freely in the full space-time bulk. The matter fields are confined to a d dimensional sub-manifold (brane) which itself can freely move in the bulk.

We note on the influence of fluctuations in the position of the brane in section 2.2. These fluctuations will lead to one-loop contributions which are disguised as purely gravitational ones.

Section 2.3 will be a short sketch of the matter Lagrangians and how we obtain the Feynman rules for the coupling between the matter fields and the graviton.

In the last section of this chapter we discuss the energy expansion of the effective field theory and determine the range of energy and parameters (masses) in which our calculations are valid.

2.1. Large Extra Dimensions

Before we start our calculation, we have to pick one scenario for large extra dimensions out of the manifold possible set-ups, e. g., [33–35, 47, 48]. The first point to address is the position of the brane—a sub-manifold with a time-like direction—to which the matter fields are confined. Depending on the model the brane's position is fixed with respect to the additional bulk space directions or the brane can move freely. We choose a model with a freely moving brane. The effects of possible brane displacements decouple at one-loop order, i. e., arise from independent Feynman diagrams. To apply our results to a set-up with a fixed brane, one simply needs to discharge the contributions from brane displacements, i. e., involving the brane tension in the course of the calculation.[1]

The next question concerns topology and geometry of the compactified dimensions. We want to work in a flat background, thus the space-time manifold has to be locally Minkowskian, i. e., $\mathbb{R}^{1,D-1}$. Hence, the d non-compact directions will span $\mathbb{R}^{1,d-1}$. Since we are only interested in local quantities, a sufficiently smooth compact manifold can always be approximated by a δ-dimensional torus with a uniform radius R. Other choices of the compactified space will only differ by their volume V_δ. By this choice, the evaluation of the sums over Kaluza-Klein states is particularly simplified. Thus, we consider gravity in a D dimensional space-time $\mathcal{M} = \mathbb{R}^{1,d-1} \times T^\delta$, where T^δ is a δ-dimensional torus with a uniform radius R.

In the following upper (lower) case Latin letters are used for D dimensional (δ dimensional compactified) indices and Greek letters for d dimensional indices with $\mu = 0, 1, ..., d-1$ and

[1]These contributions will—as we will see—cancel anyway. Thus, our results will apply for both types of models.

$i = d, ..., D$. Let us further decompose the D dimensional coordinates as

$$X^M = (x^\mu, z^i)\,.$$

The metric dynamics are governed by the Einstein-Hilbert action in D dimensions:

$$S_{\rm EH}^{(D)} = \frac{2}{\kappa_{(D)}^2} \int \mathrm{d}^D X \sqrt{|G|}\, \mathcal{R} \tag{2.1}$$

Here, $\kappa_{(D)}^2 = 32\pi / M_{(D)}^{D-2}$ is the gravitational coupling constant in D space-time dimensions with $M_{(D)}$ being the Planck mass in the bulk, $G = \det G_{MN}$ is the determinant of the metric, and $\mathcal{R}$ is the D dimensional Ricci scalar

$$\mathcal{R} = G^{MN}\mathcal{R}_{MN} \tag{2.2}$$

$$\mathcal{R}_{MN} = \mathcal{R}^A{}_{MAN} \tag{2.3}$$

$$\mathcal{R}^A{}_{BMN} = \partial_N \Gamma^A_{BM} - \partial_M \Gamma^A_{BN} + \Gamma^E_{NB}\Gamma^A_{EM} - \Gamma^E_{MB}\Gamma^A_{EN}\,, \tag{2.4}$$

with the Christoffel symbol

$$\Gamma^R_{MN} = \tfrac{1}{2} G^{RS}\left(\partial_M G_{NS} + \partial_N G_{MS} - \partial_S G_{MN}\right). \tag{2.5}$$

We expand the metric as

$$G_{MN} = \eta_{MN} + \kappa_{(D)} h_{MN} \tag{2.6}$$

around the flat D-dimensional Minkowski space-time with $\eta_{MN} = \mathrm{diag}(+, -, \ldots, -)$ in terms of the graviton field h_{MN}. The factor $\kappa_{(D)}$ is introduced in order to have the canonical normalisation of the graviton propagator, see (2.18).

We now perform the Kaluza-Klein reduction of the action by expanding the field $h_{MN}(x, z)$ which is compactified on the δ-dimensional torus T^δ in the modes

$$h_{MN}(x, z) = V_\delta^{-1/2} \sum_{\vec{n}\in\mathbb{Z}^\delta} h^{(\vec{n})}_{MN}(x) e^{i\frac{\vec{n}\cdot\vec{z}}{R}}\,. \tag{2.7}$$

We insert the mode expansion into the metric decomposition (2.6):

$$G_{MN}(x, z) = \eta_{MN} + \frac{\kappa_{(D)}}{\sqrt{V_\delta}} \sum_{\vec{n}\in\mathbb{Z}^\delta} h^{(\vec{n})}_{MN}(x) e^{i\frac{\vec{n}\cdot\vec{z}}{R}}\,. \tag{2.8}$$

We define the d dimensional gravitational coupling constant

$$\kappa_{(d)} = \frac{\kappa_{(D)}}{\sqrt{V_\delta}} \tag{2.9}$$

and the corresponding Planck mass

$$M_{(d)}^{d-2} = \frac{{\kappa_{(d)}}^2}{32\pi}\,. \tag{2.10}$$

This leads to the general form of (1.1)

$$M_{(d)}^{d-2} = M_{(D)}^{D-2} V_\delta = M_{(D)}^{D-2} (2\pi R)^\delta \tag{2.11}$$

for the brane and bulk Planck mass in $(d+\delta)$ dimensions.

It will be convenient to write the background metric η_{MN} and the graviton field h_{MN} in matrix form as

$$\eta_{MN} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & -\delta_{ij} \end{pmatrix} \qquad \text{and} \tag{2.12}$$

$$h_{MN} = \begin{pmatrix} \hat{h}_{\mu\nu} - \frac{1}{d-2}\phi\,\eta_{\mu\nu} & \frac{1}{\sqrt{2}} B_{i\,\nu} \\ \frac{1}{\sqrt{2}} B_{j\,\mu} & \phi_{ij} \end{pmatrix}, \tag{2.13}$$

with $\eta_{\mu\nu} = \mathrm{diag}(+, -, \ldots, -)$.

Here we have introduced the fields which appear in the d dimensional effective theory, i. e., the gravitensor $\hat{h}_{\mu\nu}$, graviphotons $B_{i\,\mu}$ and graviscalars ϕ_{ij} and further have used

$$\phi = \eta^{ij}\phi_{ij} = -\delta_{ij}\phi_{ij}$$

Next, we quantise the gravitational interaction using the established methods for Yang-Mills gauge theories following the standard program of the effective field theory of gravity [7, 8]. We equip the Einstein-Hilbert action (2.1) with a gauge fixing and a ghost term using the Faddeev-Popov prescription [49]:

$$S = \int \mathrm{d}^D X (\mathcal{L}_{\text{grav.}} + \mathcal{L}_{\text{ghosts}}) \tag{2.14}$$

$$\mathcal{L}_{\text{grav.}}^{(D)} = \frac{2}{\kappa_{(D)}^2} \sqrt{|G|}\,\mathcal{R} + \frac{1}{\alpha} F_N F^N. \tag{2.15}$$

F_N denotes the gauge fixing term

$$F_N = \partial^M \left(h_{MN} - \tfrac{1}{2}\eta_{MN} h \right) \tag{2.16}$$

with $h = h_M^M$ and α being the gauge parameter. In particular, we will consider here the de Donder gauge $\alpha = 1$ which leads to a particularly simple form of the graviton propagator.

The ghost Lagrangian can be derived the in usual way

$$\mathcal{L}_{\text{ghost}} = \bar{c}^M \frac{\delta F_M}{\delta \varepsilon^N} c^N \tag{2.17}$$

from the change of the gauge fixing condition under an infinitesimal diffeomorphism $X^M \to X^M + \varepsilon^M$. In our one-loop calculations the gravitational Faddeev-Popov ghosts will play no role, since in de Donder gauge they do no couple to matter field. Hence, there is no need to specify the ghost Lagrangian.

Next, we expand the bulk Lagrangian around the flat background. To determine the propagators, we need only the quadratic part in the gravitational field

$$
\begin{aligned}
\mathcal{L}^{(D)}_{\text{grav.}}\Big|_{h^2} =& \frac{2}{\kappa^2_{(D)}}\sqrt{|G|}R + \frac{1}{\alpha}F_N F^N\Big|_{h^2} \\
=& \tfrac{1}{2}\partial_A h_{MN}\left(\eta^{MR}\eta^{NS} - (1-\tfrac{1}{2\alpha})\eta^{MN}\eta^{RS}\right)\partial^A h_{RS} \\
& + (1-\tfrac{1}{\alpha})\partial_M h \partial_N h^{MN} \\
& - (1-\tfrac{1}{\alpha})\partial_N h^{MN}\partial^R h_{MR}
\end{aligned}
\tag{2.18}
$$

which simplifies in de Donder gauge to

$$
\mathcal{L}^{(D)}_{\text{grav.}}\Big|_{h^2} = \tfrac{1}{2}\partial_A h_{MN}\left(\eta^{MR}\eta^{NS} - \tfrac{1}{2}\eta^{MN}\eta^{RS}\right)\partial^A h_{RS}\,, \tag{2.19}
$$

where η_{MN} is used to raise and lower indices.

By integrating the Lagrangian (2.19) over the compactified extra coordinates, we obtain the d dimensional Lagrangian for Kaluza-Klein graviton states. The quadratic part of this Lagrangian reads:

$$
\begin{aligned}
\mathcal{L}^{(d)}_{\text{grav.}}\Big|_{h^2} = \frac{1}{2}\sum_{\vec{n}}\Big(&\partial_\alpha \hat{h}^{(\vec{n})}_{\mu\nu}\left(\eta^{\mu\rho}\eta^{\nu\sigma} - \tfrac{1}{2}\eta^{\mu\nu}\eta^{\rho\sigma}\right)\partial^\alpha \hat{h}^{(-\vec{n})}_{\rho\sigma} - \\
& - m^2_{\vec{n}}\hat{h}^{(\vec{n})}_{\mu\nu}\left(\eta^{\mu\rho}\eta^{\nu\sigma} - \tfrac{1}{2}\eta^{\mu\nu}\eta^{\rho\sigma}\right)\hat{h}^{(-\vec{n})}_{\rho\sigma} - \\
& - \partial_\alpha B^{(\vec{n})}_{i\,\mu}\partial^\alpha B^{(-\vec{n})\,\mu}_{i} + \\
& + m^2_{\vec{n}} B^{(\vec{n})}_{i\,\mu} B^{(-\vec{n})\,\mu}_{i} + \\
& + \partial_\alpha \phi^{(\vec{n})}_{ij}\left(\delta_{ik}\delta_{jl} + \tfrac{1}{d-2}\delta_{ij}\delta_{kl}\right)\partial^\alpha \phi^{(-\vec{n})}_{kl} - \\
& - m^2_{\vec{n}}\phi^{(\vec{n})}_{ij}\left(\delta_{ik}\delta_{jl} + \tfrac{1}{d-2}\delta_{ij}\delta_{kl}\right)\phi^{(-\vec{n})}_{kl}\Big),
\end{aligned}
\tag{2.20}
$$

where

$$
m^2_{\vec{n}} = \vec{n}\cdot\vec{n}/R^2 \tag{2.21}
$$

is the mass squared of the n$^{\text{th}}$ excited Kaluza-Klein graviton.

From this we can read off the Feynman rules for the propagators of the gravitational fields in de Donder gauge:

$$
\begin{aligned}
\hat{h}^{(\vec{n})}_{\alpha\beta} \quad \hat{h}^{(\vec{n}')}_{\gamma\delta} &: \quad \frac{i\delta_{\vec{n},-\vec{n}'}\,\frac{1}{2}\left(\eta_{\alpha\gamma}\eta_{\beta\delta} + \eta_{\alpha\delta}\eta_{\beta\gamma} - \frac{2}{d-2}\eta_{\alpha\beta}\eta_{\gamma\delta}\right)}{p^2 - m^2_{\vec{n}}} \\
B^{(\vec{n})}_{i\,\mu} \quad B^{(\vec{n}')}_{j\,\nu} &: \quad \frac{-i\delta_{\vec{n},-\vec{n}'}\,\delta_{ij}\eta_{\mu\nu}}{p^2 - m^2_{\vec{n}}} \\
\phi^{(\vec{n})}_{ij} \quad \phi^{(\vec{n}')}_{kl} &: \quad \frac{i\delta_{\vec{n},-\vec{n}'}\,\frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{D-2}\delta_{ij}\delta_{kl}\right)}{p^2 - m^2_{\vec{n}}}.
\end{aligned}
\tag{2.22}
$$

The degrees of freedom from the extra dimensional gravity are encoded in the gravitensor $\hat{h}^{(\vec{n})}_{\mu\nu}$, the graviphotons $B^{(\vec{n})}_{i\,\mu}$, and the graviscalars $\phi^{(\vec{n})}_{ij}$. For each type we have an infinite tower of Kaluza-Klein excitations. The propagator of the tensor part $\hat{h}^{(\vec{n})}_{\mu\nu}$ of the $(d{+}\delta)$-dimensional graviton has the same index structure as the propagator of the graviton in d dimensions without extra dimensions. Its zero mode $\hat{h}^{(0)}_{\mu\nu}$ is exactly the standard d dimensional graviton in de Donder gauge.

2.1.1. The matter brane

In contrast to the gravitons, which are moving freely in the bulk, the matter fields are confined to a d dimensional space-time manifold (a $(d-1)$-brane). In particular, we shall use the brane coordinates

$$Y^N(x^\mu) = (y^\mu(x) = x^\mu, \tfrac{1}{\sqrt{\tau}}\xi^i(x)),$$

where as discussed by Sundrum [50] the reparametrisation invariance of the $(d-1)$-brane allows to fix d of the coordinates and choose a static gauge $y^\mu = x^\mu$. The ξ^i are dynamical *branon* fields representing the transversal fluctuations of the brane forming the Goldstone scalars in d dimensions of the broken translation invariance in the δ extra dimensions. τ is the brane tension introduced here to yield a canonical normalization for the branons (see below). At this point, we discard the interesting question of how such a brane can dynamically arise as a solution of the underlying Einstein-Yang-Mills system or a more general supergravity theory related to string theory. It is worth mentioning that the perturbation in the gravitational dynamics due to the brane tension is small in the region of validity of the effective field theory, see section 2.4.

The Lagrangian of the brane matter does not depend directly on the bulk metric, but the induced metric on the brane. It can be express in terms of the the bulk metric and the brane's position:

$$\begin{aligned}
g_{\mu\nu}(x) &= \frac{\partial Y^M}{\partial x^\mu}\frac{\partial Y^N}{\partial x^\nu} G_{MN}(Y(x)) \\
&= G_{\mu\nu}(x, \tfrac{\xi(x)}{\sqrt{\tau}}) + \frac{1}{\sqrt{\tau}}\Big(\partial_\mu \xi^i G_{i\nu}(x, \tfrac{\xi(x)}{\sqrt{\tau}}) + \partial_\nu \xi^i G_{\mu i}(x, \tfrac{\xi(x)}{\sqrt{\tau}})\Big) + \\
&\quad + \frac{1}{\tau}\Big(\partial_\mu \xi^i \partial_\nu \xi^j G_{ij}(x, \tfrac{\xi(x)}{\sqrt{\tau}})\Big) \\
&= G_{\mu\nu}(x,0) \\
&\quad + \frac{1}{\sqrt{\tau}}\Big(\xi^i \partial_i G_{\mu\nu}(x,0) + \partial_\mu \xi^i G_{i\nu}(x,0) + \partial_\nu \xi^i G_{\mu i}(x,0)\Big) + \\
&\quad + \frac{1}{\tau}(\tfrac{1}{2}\xi^i\xi^j\partial_i\partial_j G_{\mu\nu}(x,0) + \xi^i \partial_\mu \xi^j \partial_i G_{j\nu}(x,0) + \\
&\qquad + \xi^i \partial_\nu \xi^j \partial_i G_{\mu j}(x,0) + \partial_\mu \xi^i \partial_\nu \xi^j G_{ij}(x,0)) + \mathcal{O}(\tau^{-3/2})\,.
\end{aligned} \tag{2.23}$$

Again, it can be expanded around the flat background

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa\, \tilde{h}_{\mu\nu}, \tag{2.24}$$

using now the d-dimensional gravitational constant $\kappa \equiv \kappa_{(d)} = \kappa_{(D)}/(2\pi R)^{\delta/2}$. The metric fluctuation $\tilde{h}$ has to be expressed in terms of the branon ξ and the D-dimensional graviton

h_{MN}. From the mode expansion of the bulk metric (2.8) and (2.23) we obtain

$$
\begin{aligned}
\tilde{h}_{\mu\nu} = & -\frac{1}{\kappa\tau}\delta_{ij}\partial_\mu\xi^i\partial_\nu\xi^j + \\
& + \sum_{\vec{n}}\Big(\hat{h}^{(\vec{n})}_{\mu\nu} - \tfrac{1}{d-2}\eta_{\mu\nu}\phi^{(\vec{n})} + \frac{1}{\sqrt{\tau}}\Big(\tfrac{i}{R}n_i\xi^i\hat{h}^{(\vec{n})}_{\mu\nu} - \tfrac{i}{R(d-2)}\eta_{\mu\nu}n_i\xi^i\phi^{(\vec{n})} + \\
& \qquad + \tfrac{1}{\sqrt{2}}\partial_\mu\xi^i B^{(\vec{n})}_{i\,\nu} + \tfrac{1}{\sqrt{2}}\partial_\nu\xi^i B^{(\vec{n})}_{i\,\mu}\Big)\Big) + \mathcal{O}\left(\frac{1}{\tau}\right).
\end{aligned}
\tag{2.25}
$$

The omitted $\mathcal{O}\left(\frac{1}{\tau}\right)$ terms are at least cubic in the fields and are not necessary for our calculations. This expression will be used in the following to obtain the Feynman rules for the coupling between the matter fields and the graviton and branon fields.

2.2. Interaction with branons

2.2.1. Branon propagator and graviton–branon mixing

Let us now consider the brane Lagrangian

$$
\mathcal{L}_{\text{brane}} = -\sqrt{|g|}\tau + \mathcal{L}_{\text{matter}}.
$$

The matter Lagrangian $\mathcal{L}_{\text{matter}}$ will be specified in section 2.3. To determine the propagator of the branon fields, we expand the first term in $\mathcal{L}_{\text{brane}}$ up to quadratic order in graviton and branon fields which yields

$$
\begin{aligned}
-\tau\sqrt{|g|} = & -\tau\Big(1 + \frac{\kappa}{2}\sum_{\vec{n}}\left(\hat{h}^{(\vec{n})} - \tfrac{d}{d-2}\phi^{(\vec{n})}\right) + \\
& + \frac{\kappa^2}{8}\sum_{\vec{n}}\sum_{\vec{m}}\left(\hat{h}^{(\vec{n})}\hat{h}^{(\vec{m})} - 2\hat{h}^{(\vec{n})\,\alpha\beta}\hat{h}^{(\vec{m})}_{\alpha\beta} - 2\hat{h}^{(\vec{n})}\phi^{(\vec{m})} + \tfrac{d}{d-2}\phi^{(\vec{n})}\phi^{(\vec{m})}\right)\Big) + \\
& + \frac{1}{2}\delta_{ij}\partial^\mu\xi^i\partial_\mu\xi^j - \\
& - \frac{\kappa\sqrt{\tau}}{2}\sum_{\vec{n}}\left(\tfrac{i}{R}n_i\xi^i\hat{h}^{(\vec{n})} - \tfrac{id}{R(d-2)}n_i\xi^i\phi^{(\vec{n})} + \sqrt{2}B^{(\vec{n})}_{i\,\mu}\partial^\mu\xi^i\right) + \mathcal{L}_{\text{interaction}}\,.
\end{aligned}
\tag{2.26}
$$

Note that equation (2.26) contains terms linear in $\hat{h}$ and ϕ, because the massive brane is a source of gravity. These terms reflect the off-shell nature of the metric expansion, but they can be neglected for the regime under consideration, see section 2.4. Furthermore we find a kinetic term for the massless branons, graviton-branon mixing terms as well as interaction terms. The corresponding Feynman rules are

$$
\begin{aligned}
& \xi^i \quad \xi^j : \quad \frac{i\delta^{ij}}{p^2}\,, \\
& \hat{h}^{(\vec{n})}_{\mu\nu} \quad \xi^i : \quad \frac{\kappa\sqrt{\tau}}{2}\eta_{\mu\nu}\frac{n_i}{R}\,, \\
& B^{(\vec{n})}_{j\,\mu} \quad \xi^i : \quad \frac{\kappa\sqrt{\tau}}{\sqrt{2}}\delta^j_i p_\mu\,, \\
\text{and} \quad & \phi^{(\vec{n})}_{kl} \quad \xi^i : \quad \frac{\kappa\sqrt{\tau}}{2}\frac{d}{d-2}\delta_{kl}\frac{n_i}{R}\,,
\end{aligned}
\tag{2.27}
$$

where p_μ is the incoming momentum of the graviphoton $B^{(\vec{n})}_{j\,\mu}$ and n_i the i'th mode number of the Kaluza-Klein field.

2.2.2. Interactions of branons and matter fields

The branon fields interact with the brane matter via the induced metric (2.23). The first order couplings of the branon and gravitational fields to brane fields are mediated by

$$\mathcal{L}^{(\kappa)} = -\frac{\kappa}{2} T^{\mu\nu} \tilde{h}_{\mu\nu}$$

and shown in Figure 2.1. Here $T^{\mu\nu}$ is the energy-momentum tensor of the brane matter fields. The vertices involving branons are quadratic in the gravitational and branon fields. The one-loop amplitudes we want to calculate, i. e., whose without external graviton and branon fields, do not involve higher order couplings with the branons. Starting from the second order coupling $\sim \kappa^2 \frac{\delta^2 \mathcal{L}_{\text{matter}}}{\delta g_{\mu\nu} \delta g_{\rho\sigma}} \tilde{h}_{\mu\nu} \tilde{h}_{\rho\sigma}$, it is sufficient for our calculations to substitute $\tilde{h}_{\mu\nu} = \sum_{\vec{n}} \left(\hat{h}^{(\vec{n})}_{\mu\nu} - \frac{1}{d-2} \eta_{\mu\nu} \phi^{(\vec{n})} \right)$ and discharge the branons.

Furthermore, we can now see how the branons might influence the one-loop corrections at order κ^2: The graviton–branon mixing (2.27) operators are proportional to $\kappa\sqrt{\tau}$ and the branon–matter interactions come with a factor of $1/\sqrt{\tau}$ for each branon. It is easy to construct one-loop Feynman diagrams in which the brane tension cancels. Thus, the branons, i. e., the fact that the brane position is not fixed, will lead to one-loop contributions which are disguised as purely gravitational ones. We will discuss the $\mathcal{O}(\kappa^2)$ branon contributions in section 5.1.

2.3. Interactions with matter fields

We are interested in the gravitational corrections to couplings of the Standard Model. All matter fields are in our set-up confined to the $(d-1)$-brane. Consequently, the matter Lagrangians only depend on the induced metric (2.23). The gravitational fields will enter the calculations via the induced metric fluctuations $\tilde{h}_{\mu\nu}$ (2.25).

In order to obtain the Feynman rules, we need to express the brane Lagrangians in terms of the matter and gravitational fields. For sake of readability, we only give the κ expansion of the matter Lagrangians in terms of the metric fluctuations $\tilde{h}_{\mu\nu}$. After inserting equation (2.25) into the expanded Lagrangian, it is straight forward to obtain the Feynman rules for the vertices. Since these are quite unhandy and contain no additional information, we will give no explicit formula in this thesis. We used FORM to derive the Feynman rules which are only an intermediate step in the FORM scripts. The scripts we wrote will yield the fully contracted results of the one-loop diagrams. We will give an example of our FORM scripts in section 4.5.

2.3.1. Scalars

We start with the Lagrangian of the minimally coupled scalar field:

$$\mathcal{L}_s = \sqrt{|g|} g^{\mu\nu} (\mathcal{D}_\mu \phi)^\dagger \mathcal{D}_\nu \phi - \sqrt{|g|} m_\phi^2 \phi^\dagger \phi \qquad \text{with} \qquad \mathcal{D}_\mu = \partial_\mu - igA_\mu \tag{2.28}$$

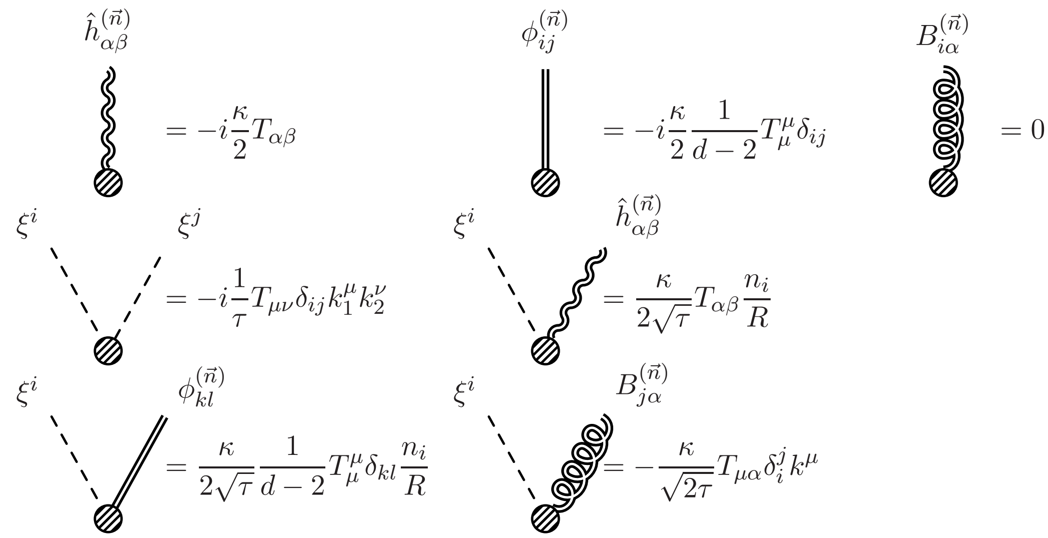


Figure 2.1.: Leading order vertices of graviton–matter and branon–matter couplings; k_1, k_2, k are the incoming momenta of the branons.

and expand the Lagrangian in orders of the gravitational coupling κ

$$\begin{aligned}\mathcal{L}_s = (\mathcal{D}_\mu\phi)^\dagger\mathcal{D}^\mu\phi + \kappa\left[\left(\tfrac{1}{2}\eta^{\mu\nu}\tilde{h} - \tilde{h}^{\mu\nu}\right)(\mathcal{D}_\mu\phi)^\dagger\mathcal{D}_\nu\phi - \tfrac{1}{2}\tilde{h}m_\phi^2\phi^\dagger\phi\right] \\ +\kappa^2\Big[\left(\tfrac{1}{8}(\tilde{h}^2 - 2\tilde{h}^{\alpha\beta}\tilde{h}_{\alpha\beta})\eta^{\mu\nu} + \tilde{h}^{\mu\alpha}\tilde{h}^{\nu}_{\alpha} - \tfrac{1}{2}\tilde{h}\tilde{h}^{\mu\nu}\right)(\mathcal{D}_\mu\phi)^\dagger\mathcal{D}_\nu\phi \\ - \tfrac{1}{8}(\tilde{h}^2 - 2\tilde{h}^{\alpha\beta}\tilde{h}_{\alpha\beta})m_\phi^2\phi^\dagger\phi\Big] + \mathcal{O}(\kappa^3)\,.\end{aligned} \tag{2.29}$$

The Feynman rules for its coupling to the gravitational fields are obtained by inserting formula (2.25) for $\tilde{h}_{\mu\nu}$.

2.3.2. Fermions

In order to calculate the gravitationally induced renormalisation of a spinor field and its couplings, we consider the Lagrangian of a Dirac fermion on the $(d-1)$-brane

$$\mathcal{L}_f = \sqrt{|g|}\,\overline{\psi}\,(i\not{\mathcal{D}} - m_{\psi})\,\psi \tag{2.30}$$

and write the covariant derivative for the spinor explicitly:

$$\not{\mathcal{D}} = \gamma^a e_a^\mu(\partial_\mu - i\tfrac{1}{2}S_{ab}\omega_\mu^{ab} - igA_\mu)\,. \tag{2.31}$$

with the Dirac matrices γ^a and the spinor representation of the Lorentz algebra $S_{ab} = \frac{i}{4}[\gamma_a, \gamma_b]$. One notices that instead of the metric, the vielbein e_μ^a is the geometric quantity which couples to the spinor. The fermion Lagrangian depends on the inverse vielbein e_a^μ and the spin connection

$$\omega_\mu^{ab} = e_\nu^a \nabla_\mu e^{b\nu} = e_\nu^a \partial_\mu e^{b\nu} + e_\nu^a e^{b\rho}\Gamma^\rho_{\nu\mu}\,.$$

The vielbein field defines a local Lorentz frame (denoted by Latin indices $a, b, \dots$) at each space-time point and are connected to the metric by the relation

$$e_\mu^a(x) e_\nu^b(x)\eta^{ab} = g_{\mu\nu}(x). \tag{2.32}$$

The local Lorentz frame has an additional $O(1, d-1)$ symmetry which leads to an additional gauge freedom in the vielbeins. At first sight, the Einstein-Dirac system seems to be fundamentally different to the coupling of bosonic fields to gravity [3]. However, it is long known that the Lorentz symmetry can be fixed at classical level [51, 52] without losing information in the quantum theory. The flat background decomposition of the induced metric on the brane (2.24) yields

$$e_\mu^a = \delta_\mu^a + \tfrac{\kappa}{2}\tilde{h}_\mu^a - \tfrac{\kappa^2}{8}\tilde{h}_\mu^\nu \tilde{h}_\nu^a + \mathcal{O}(\kappa^3) \tag{2.33}$$

$$e_a^\mu = \delta_a^\mu - \tfrac{\kappa}{2}\tilde{h}_a^\mu + \tfrac{3\kappa^2}{8}\tilde{h}_\nu^\mu \tilde{h}_a^\nu + \mathcal{O}(\kappa^3) \tag{2.34}$$

for the vielbein and inverse vielbein.

Starting from (2.30) we proceed by expanding the metric dependent quantities around flat space. The expansion of the Lagrangian to the needed order is given by

$$\begin{aligned}\mathcal{L}_f = \overline{\psi}(i\not{\mathcal{D}} - m_\psi)\psi &+ \tfrac{\kappa}{2}\,\overline{\psi}\Big[\tilde{h}(i\not{\mathcal{D}} - m_\psi) - i\tilde{\not{h}}^\mu \mathcal{D}_\mu - \tfrac{i}{2}(\partial_a \tilde{\not{h}}_b)\gamma^{ab}\Big]\psi \\ &+ \tfrac{\kappa^2}{8}\,\overline{\psi}\Big[(\tilde{h}^2 - 2\tilde{h}^{\alpha\beta}\tilde{h}_{\alpha\beta})(i\not{\mathcal{D}} - m_\psi) + i(3\tilde{h}^\mu_\rho \tilde{\not{h}}^\rho - 2h\tilde{\not{h}}^\mu)\mathcal{D}_\mu - i\tilde{h}(\partial_a \tilde{\not{h}}_b)\gamma^{ab} \\ &\qquad + i\tilde{h}^\mu_b(\partial_a \tilde{\not{h}}_\mu - \partial_\mu \tilde{\not{h}}_a + \tfrac{1}{2}\not{\partial}\tilde{h}_{\mu a})\gamma^{ab} + i\tilde{\not{h}}^\mu(\partial_a \tilde{h}_{\mu b})\gamma^{ac}\Big]\psi + \mathcal{O}(\kappa^3)\end{aligned} \tag{2.35}$$

with $\tilde{h} = \tilde{h}^\mu_\mu$, $\gamma^{ab} = \gamma^{[a}\gamma^{b]}$, and $\mathcal{D}_\mu = \partial_\mu - igA_\mu$.

2.3.3. Gauge Fields

The Yang-Mills part of the brane Lagrangian is

$$\mathcal{L}_{\mathrm{YM}} = -\tfrac{1}{2}\sqrt{|g|}\, g^{\mu\rho} g^{\nu\sigma}\, \mathrm{tr}\{F_{\mu\nu}F_{\rho\sigma}\}, \tag{2.36}$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \tag{2.37}$$

the gauge field strength, $A_\mu = A_\mu^a T^a$ the gauge field, T^a the Lie algebra generators of the gauge group and g the gauge boson coupling. Since $F_{\mu\nu}$ is antisymmetric, the Christoffel connections arising from the space-time covariant derivatives ∇_μ cancel against each other, hence the covariant derivatives can be replaced here by ordinary derivatives ∂_μ.

The expansion of (2.36) in the graviton field yields

$$\begin{aligned}\mathcal{L}_{\mathrm{YM}} = -\tfrac{1}{2}\Big[\eta^{\mu\rho}\eta^{\nu\sigma} - \kappa\left(\tilde{h}^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\rho}\tilde{h}^{\nu\sigma} - \tfrac{1}{2}\tilde{h}\,\eta^{\mu\rho}\eta^{\nu\sigma}\right) + \kappa^2\Big(\tilde{h}^{\mu\rho}\tilde{h}^{\nu\sigma} + \tilde{h}^{\mu\alpha}\tilde{h}^{\rho}_{\alpha}\eta^{\nu\sigma}\\ + \eta^{\mu\rho}\tilde{h}^{\nu\alpha}\tilde{h}^{\sigma}_{\alpha} - \tfrac{1}{2}\tilde{h}\,(\tilde{h}^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\rho}\tilde{h}^{\nu\sigma})\\ + \tfrac{1}{8}(\tilde{h}^2 - 2\tilde{h}^{\alpha\beta}\tilde{h}_{\alpha\beta})\eta^{\mu\rho}\eta^{\nu\sigma}\Big)\Big]\,\mathrm{tr}\{F_{\mu\nu}F_{\rho\sigma}\} + \mathcal{O}(\kappa^3)\,.\end{aligned} \tag{2.38}$$

2.4. Validity of the energy expansion

Before starting the calculations, we have to check whether the chosen scenario is able to give meaningful results. The effective field theory computation implies in addition to the perturbative expansion also an expansion in energy ratios which accordingly have to be small.

In our investigation of possible one-loop contributions from gravitons and branons we focus only on gravity effects $\mathcal{O}(\kappa^2)$ and not on effects of the brane tension τ, i. e., neglecting both terms $\mathcal{O}(\tau)$ and $\mathcal{O}(\tau^{-1})$. In this section we show that an energy hierarchy allowing this ansatz exists, due to limited number of combinations of energy scales appearing in the expansion. The various energy scales of the effective field theory are:

The gravitational coupling	$\kappa_{(D)} \simeq M_{(D)}^{\frac{2-D}{2}}$
The brane tension	$\tau = M_\tau^d$
The particle energy and masses	$E \sim m_{\{\varphi,\psi\}}$
The volume of the compactified Dimensions	$M_V = (2\pi R)^{-1}$.

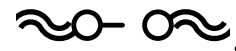
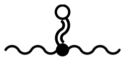
For the effective theory of gravity to be valid, we have the following requirements:

1. Of course, the expansion parameter itself must be small. Keeping the number of incoming and outgoing particles, fixed the expansion parameter is

$$\kappa_{(D)} E^{\frac{D-2}{2}} = \left(\frac{E}{M_{(D)}}\right)^{\frac{D-2}{2}} \ll 1 \quad \Longrightarrow \quad \frac{E}{M_{(D)}} \ll 1\,. \tag{2.39}$$

2. In our consideration we neglected higher branon corrections, e. g. , which are at least $\mathcal{O}(\tau^{-1})$. Thus we need

$$\frac{E^d}{\tau} = \left(\frac{E}{M_\tau}\right)^d \ll 1 \quad \Longrightarrow \quad \frac{E}{M_\tau} \ll 1\,. \tag{2.40}$$

3. As a consequence of (2.26) the effective theory includes tree-level mixing between Kaluza-Klein graviton states, direct  and indirect via branons , as well as tadpoles . All of these effects are of order $\mathcal{O}(\kappa^2_{(D)}\tau)$. Neglecting their effect requires

$$\kappa^2_{(D)}\tau = \frac{M_\tau^d}{M_{(D)}^{D-2}} = M^2_{(D)}\frac{M_\tau^d}{M_{(D)}^D} \ll E^{2-D+d} = E^{2-\delta}\,. \tag{2.41}$$

To find a consistent energy hierarchy in which the mixing can be ignored, we have to consider the following cases:

a) $\delta > 2$, assuming $M_\tau \gg M_{(D)}$ (2.41) is equivalent to:

$$E^{\delta-2} \ll \frac{M_{(D)}^{D-2}}{M_\tau^d} \overset{M_{(D)}\ll M_\tau}{\ll} M_\tau^{D-2-d} = M_\tau^{\delta-2},$$

i. e., $M_\tau \gg E$, in agreement with the requirement (2.41).
The assumption $M_\tau \gg M_{(D)}$ is feasible, since only the brane density $M_\tau^d\rho^{-\delta}$, with ρ being the thickness, is bounded by the D dimensional Planck scale $M_{(D)}^D$.

b) In the case $\delta = 2$ (2.41) simplifies to

$$M_\tau \ll M_{(D)}\,.$$

c) Finally, if $\delta = 1$ (2.41) is equivalent to

$$M_\tau^d \ll E M_{(D)}^{d-1} \ll M_{(D)}$$

but it should still be $E^1 M_{(D)}^{d-1} \gg E^d$, so that (2.40) can be satisfied.

Thus, we find depending on the number of extra dimensions three different energy hierarchies:

$$\text{for } \delta > 2 \qquad E \ll M_{(D)} \ll M_\tau\,, \tag{2.42}$$

$$\text{for } \delta = 2 \qquad E \ll M_\tau \ll M_{(D)} \tag{2.43}$$

$$\text{and for } \delta = 1 \qquad E \ll M_\tau \ll E^{1/d} M_{(D)}^{(d-1)/d} \ll M_{(D)} \tag{2.44}$$

in which our perturbative considerations are meaningful.

3. Higher Derivative Operators

One of the aspects of the non-renormalisable nature of perturbatively quantised gravity is the necessity to include terms in the effective field theory Lagrangian whose couplings have negative mass dimension. These terms are needed to absorb all divergences in the renormalisation process.

The one-loop contributions of order κ^2 corresponds to terms with up to two additional space-time derivatives. Since we are interested in fields which are charged under gauge symmetries these do not commute even in a flat Minkowskian background.

Our motivation to have a closer look at the gauge covariant higher derivative terms is the possible connection between gravitational renormalisation and the so called Lee-Wick Standard Model (LWSM) first pointed out in [44]. We start in the first section with a short sketch of the Lee-Wick extension of the Standard Model. In the main part of this chapter we will derive a basis for the higher derivative operators and give the corresponding Feynman rules.

3.1. The Lee-Wick Standard Model

In [38, 39], Lee and Wick studied the consequences of the assumption that the modification of the photon propagator in the Pauli-Villars regularisation [40] of quantum electrodynamics might correspond to a physical degree of freedom. The modification of the Maxwell action by a higher derivative operator leads to an additional pole of the photon propagator. This second pole can be interpreted as an new massive vector field, but with the wrong sign of the propagator. Consequently, the particle is a ghost, as it leads to negative norm states in the Hilbert space and thus violates unitarity. These problems appear to be solvable at low loop orders and have been extensively discussed in the literature, e. g., in [38, 39, 42, 43], but no proof at arbitrary order in perturbation theory exists. However, no examples of unitarity violation in Lee-Wick theories are known. For our considerations, the unitarity problem plays no role, since we only calculate in an effective field theory with an physical energy cut-off and the mass of the ghost field is expected to be above the cut-off.

In 2008, Grinstein, O'Connell, and Wise [37] extended the Standard Model to include higher derivative terms for each particle:

$$\begin{aligned} &\tfrac{1}{M_A^2}\,\mathrm{tr}\{(\mathcal{D}^\mu F_{\mu\nu})^2\} && \text{for gauge fields,} \\ &\tfrac{1}{M_\phi^2}(\mathcal{D}^2\phi)^\dagger(\mathcal{D}^2\phi) && \text{for scalars (Higgs), and} \\ &\tfrac{i}{M_\psi^2}\overline{\psi}\not{\mathcal{D}}^3\psi && \text{for fermions.} \end{aligned} \tag{3.1}$$

These so called Lee-Wick terms have the same properties as the Pauli-Villars regulator in Lee and Wick's higher derivative version of quantum electrodynamics. They allow for an equivalent formulation of the theory containing additional massive ghosts but only operators of dimension four[1] or less.

To illustrate the appearance of massive ghost fields in the LWSM, let us have a look at the higher derivative Lagrangian of a scalar field:

$$\mathcal{L} = (\mathcal{D}_\mu \phi)^\dagger \mathcal{D}^\mu \phi - \frac{1}{M_\phi^2} (\mathcal{D}^2 \phi)^\dagger \mathcal{D}^2 \phi - V(|\phi|)$$

Introducing an auxiliary field $\tilde{\phi}$, it can be rewritten without higher space-time derivatives:

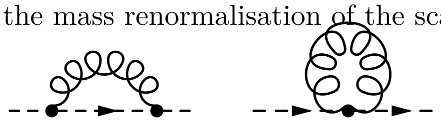
$$\mathcal{L} = (\mathcal{D}_\mu \phi)^\dagger \mathcal{D}^\mu \phi + M_\phi^2 (\tilde{\phi})^\dagger \tilde{\phi} + (\mathcal{D}_\mu \phi)^\dagger \mathcal{D}^\mu \tilde{\phi} + (\mathcal{D}_\mu \tilde{\phi})^\dagger \mathcal{D}^\mu \phi - V(|\phi|)\,.$$

After the linear field redefinition $\phi = \hat{\phi} - \tilde{\phi}$:

$$\mathcal{L} = \mathcal{D}_\mu \hat{\phi} \mathcal{D}^\mu \hat{\phi} - (\mathcal{D}_\mu \tilde{\phi})^\dagger \mathcal{D}^\mu \tilde{\phi} + M_\phi^2 (\tilde{\phi})^\dagger \tilde{\phi} - V(|\hat{\phi} - \tilde{\phi}|)$$

we obtain a theory of two scalars $\hat{\phi}$ and $\tilde{\phi}$. The first is the standard scalar field, the second is the massive Lee-Wick ghost. This reformulation is only possible for the special Lee-Wick higher derivative operators. Other dimension-six (dimension-d+2) operators like $(\mathcal{D}_\mu \phi)^\dagger F^{\mu\nu} \mathcal{D}_\nu \phi$ cannot be removed from the theory. Such terms will lead to a growth of the amplitudes with energy [41], thus violating perturbative unitarity. That is the reason why it is crucial for the LWSM that the operators (3.1) are the *only* dimension-six terms of the theory.

In contrast to the higher derivative QED, Grinstein's Lee Wick Standard Model (LWSM) is not finite, but is still free of quadratic divergences. The Lee-Wick ghosts act in a similar way as the super-symmetric partners in the MSSM. For example the contributions of the gauge fields (curly lines) to the mass renormalisation of the scalar field



are accompanied by the corresponding contributions of the Lee-Wick gauge fields (double lines):

The opposite sign of the propagators leads to the cancellation of the quadratic divergences. Therefore, the LWSM could be a solution to the hierarchy puzzle.

The Lee-Wick masses have to be introduced as phenomenological parameters in the LWSM. On the other hand, in a perturbative quantum gravity set-up which includes coupling to the matter fields, operators with additional derivatives appear naturally as counterterms. However, in the case of charged fields multiple linearly independent higher derivative operators exist, see next section; and all of which might need to be renormalised. If all necessary higher derivative counterterms are *exactly* of the form given in (3.1) this might hint towards a connection between the LWSM and quantum gravity.

The first to notice this possibility were Wu and Zhong in [44]. They analysed the Einstein-Maxwell system and found that the only gravity induced counterterm is of the form $\partial^\mu F_{\mu\rho} \partial_\nu F^{\nu\rho}$[2].

[1] The LWSM is formulated in four space-time dimension. For our calculations with matter on a $d-1$ brane these operators have mass dimension d.

[2] Note, that this is the only allowed higher derivative operator for Abelian gauge fields. However, in our simultaneous work [15, 16] we obtained the same result for the Einstein-Yang-Mills theory.

Later, the same authors [45] claimed that a large extra dimension model provides a mechanism for the emergence of Lee-Wick partners, with masses in the TeV scale, for all particles. They base their arguments on the higher derivative counterterms that appear in the one-loop renormalisation of this theory, which according to their results are given by the Lee-Wick terms (3.1). However, they only calculated two-point functions which alone do not determine the higher derivative counterterms, as we will show explicitly. Consequently, the natural question whether or not the fermionic and scalar higher derivative counterterms of Einstein-Yang-Mills theory are also given by the corresponding Lee-Wick terms (3.1) could not be answered by their calculation.

These claims motivated our work [53] in which we calculate the gauge covariant higher derivative terms renormalised by quantum gravity.

3.2. Bases of Higher Derivative Operators

Since gravity is perturbatively non-renormalisable, we need counterterms with additional space-time derivatives to renormalise the divergences from the one-loop gravitational corrections. For charged fields, including non-Abelian gauge bosons, these derivatives do not commute even in the flat Minikowski background to which we constrain ourselves in our considerations.

In the following, we acquire the bases of the space of higher derivative operators for each field and give the Feynman rules for the operators contributions to the two-point functions and coupling to gauge bosons.

3.2.1. Four Derivative Operators for Charged Scalars

In order to find a basis for the operators with two additional covariant derivatives, we first write down the generic operator with four derivatives acting on the scalar field

$$\phi^\dagger \mathcal{D}_\mu \mathcal{D}_\nu \mathcal{D}_\rho \mathcal{D}_\sigma \phi$$

and take all possible contraction of the space-time indices

$$\phi^\dagger \mathcal{D}_\mu \mathcal{D}^\mu \mathcal{D}_\rho \mathcal{D}^\rho \phi, \quad \phi^\dagger \mathcal{D}_\mu \mathcal{D}_\nu \mathcal{D}^\mu \mathcal{D}^\nu \phi, \quad \phi^\dagger \mathcal{D}_\mu \mathcal{D}_\nu \mathcal{D}^\nu \mathcal{D}^\mu \phi\,.$$

These can be rewritten using the fieldstrength tensor $[\mathcal{D}_\mu, \mathcal{D}_\nu] = igF_{\mu\nu}$:

$$\begin{aligned} \phi^\dagger \mathcal{D}_\mu \mathcal{D}_\nu \mathcal{D}^\nu \mathcal{D}^\mu \phi &= \phi^\dagger \left(\mathcal{D}_\mu \mathcal{D}_\nu \mathcal{D}^\mu \mathcal{D}^\nu + ig \mathcal{D}_\mu \mathcal{D}_\nu F^{\mu\nu}\right) \phi \\ &= \phi^\dagger \left(\mathcal{D}_\mu \mathcal{D}_\nu \mathcal{D}^\mu \mathcal{D}^\nu - g^2 F_{\mu\nu} F^{\mu\nu}\right) \phi \\ \text{and} \quad \phi^\dagger \mathcal{D}_\mu \mathcal{D}_\nu \mathcal{D}^\mu \mathcal{D}^\nu \phi &= \phi^\dagger \left(\mathcal{D}_\mu \mathcal{D}^\mu \mathcal{D}_\nu \mathcal{D}^\nu + ig \mathcal{D}_\mu F_{\nu\mu} \mathcal{D}^\nu\right) \phi \end{aligned}$$

We are only interested in the operator as a counterterm in the action, i. e., integrated over the space-time. So, we can further integrate by parts and discard total derivatives. In the end, we get as a basis

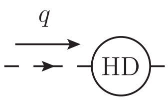
$$(\mathcal{D}^2\phi)^\dagger \mathcal{D}^2 \phi\,, \qquad ig(\mathcal{D}_\mu \phi)^\dagger F^{\mu\nu} \mathcal{D}_\nu \phi \quad \text{and} \quad g^2 \phi^\dagger F^{\mu\nu} F_{\mu\nu} \phi\,. \tag{3.2}$$

The above basis illustrates well the flaw in the discussion by Wu and Zhong [45]. The second and third term only appear in the coupling of the scalar to at least one gauge boson, respectively two gauge bosons. The scalar two-point function is not affected by them at tree level.

If we add the higher derivative terms for the scalar field in the form

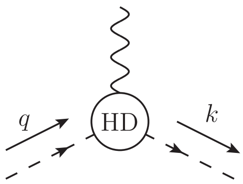
$$\mathcal{L}_s^{\mathrm{HD}} = s_1(\mathcal{D}^2\phi)^\dagger\mathcal{D}^2\phi + s_2 gi(\mathcal{D}_\mu\phi)^\dagger F^{\mu\nu}\mathcal{D}_\nu\phi + s_3 g^2\phi^\dagger F^{\mu\nu}F_{\mu\nu}\phi \tag{3.3}$$

to the usual scalar Lagrangian, these contribute to the two-point function as



$$= is_1 q^4 \tag{3.4}$$

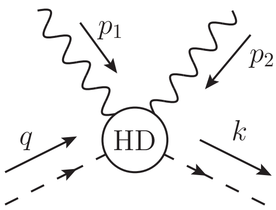
and to the coupling of the scalar to one gluon as



$$\begin{aligned} &= igt^a\big(q^\mu(s_1(q^2+k^2)+s_2(k^2-q\cdot k)) \\ &\qquad + k^\mu(s_1(k^2+q^2)+s_2(q^2-k\cdot q))\big)\,. \end{aligned} \tag{3.5}$$

In our calculation of the gravitational one-loop correction to the coupling of the scalar field to two gauge bosons, we will restrict ourselves to the *Abelian* case. We do so for a technical reason: The gluon self-interaction vertex would enlarge the number of one-loop diagrams, see Figure 5.3, of the scalar–two-gauge-bosons interaction, without yielding more information. From the calculation in an Abelian gauge theory we will still be able to read of the full result for the non-Abelian case: At one-loop level the gravitational corrections do not depend on the internal gauge group symmetries. Thus, the renormalisation of the higher derivative operators (3.2) is the same in both cases.

the Feynman rule for the higher derivative coupling of the scalar to two *photons* is



$$\begin{aligned} &= ie^2\Big(2s_1(\eta^{\mu\nu}(q^2+k^2) - p_1^\nu p_2^\mu + (q+k)^\mu(q+k)^\nu) \\ &\quad + s_2(\eta^{\mu\nu}(p_1+p_2)^2 - (p_1+p_2)^\mu(p_1+p_2)^\nu) \\ &\quad + 4s_3(p_1\cdot p_2\,\eta^{\mu\nu} - p_1^\nu p_2^\mu)\Big)\,. \end{aligned} \tag{3.6}$$

3.2.2. Three and Two Derivative Operators for Dirac Fermions

Again, we start with a generic operator with two additional derivatives, i. e., three derivatives for the fermion. In order to get an even number of space-time indices, we have to multiply the derivatives with one or three Dirac matrices:

$$i\overline{\psi}\mathcal{D}_\mu\mathcal{D}_\nu\mathcal{D}_\rho\gamma^\alpha\psi, \qquad i\overline{\psi}\mathcal{D}_\mu\mathcal{D}_\nu\mathcal{D}_\rho\gamma^\alpha\gamma^\beta\gamma^\delta\psi.$$

Now, we contract the space-time indices. The operators with one Dirac matrix are

$$i\overline{\psi}\mathcal{D}_\mu\mathcal{D}^2\gamma^\mu\psi\,, \qquad i\overline{\psi}\mathcal{D}^2\mathcal{D}_\mu\gamma^\mu\psi \quad \text{and} \quad i\overline{\psi}\mathcal{D}_\nu\mathcal{D}_\mu\mathcal{D}^\nu\gamma^\mu\psi\,. \tag{3.7}$$

The ordering of the Dirac matrices in the three γ operators is not relevant, since we can always rearrange the matrices using the Dirac algebra

$$\gamma^\nu\gamma^\mu = -\gamma^\mu\gamma^\nu + \{\gamma^\nu, \gamma^\mu\} = -\gamma^\mu\gamma^\nu + 2\eta^{\mu\nu}$$

up to terms with only one γ. The latter are already included in (3.7). Further, we can obviously ignore all index combinations where two γ's are contracted: $\gamma^\mu\gamma_\mu = d$ will again yield an operator proportional to one of (3.7). So, only one higher derivative operator with three Dirac matrices has to be included in the basis, e. g.,

$$i\overline{\psi}\mathcal{D}_\mu\mathcal{D}_\nu\mathcal{D}_\rho\gamma^\mu\gamma^\nu\gamma^\rho\psi\,.$$

To span the four dimensional space of the three derivative operators for the Dirac fermion we use the following basis:

$$i\overline{\psi}\not{\mathcal{D}}\not{\mathcal{D}}\not{\mathcal{D}}\psi\,, \qquad i\overline{\psi}\not{\mathcal{D}}\mathcal{D}^2\psi\,, \qquad i\overline{\psi}\mathcal{D}^2\not{\mathcal{D}}\psi, \quad \text{and} \quad i\overline{\psi}\mathcal{D}_\mu\not{\mathcal{D}}\mathcal{D}^\mu\psi\,. \tag{3.8}$$

The fermion mass appears linearly in the Lagrangian, in contrast to the scalar mass which only appears quadratically, i. e., as m_ϕ^2. Consequently, not the squared mass m_ψ^2 (with mass dimension two), but the mass m_ψ itself is a parameter. This is a significant difference in perturbation theory. The expansion of the renormalisation group to first order will of course never yield general functions of the bare parameters, i. e., $\sqrt{m_\phi^2}$ will not appear in our calculation. Thus in contrast to the scalar case, the gravitational contributions will also renormalise fermionic operators with only one additional derivative. The space of these operators is two dimensional and we choose as a basis

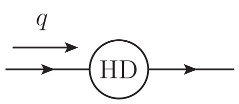
$$\overline{\psi}\mathcal{D}^2\psi \quad \text{and} \quad ig\,\overline{\psi}F_{\mu\nu}\gamma^{\mu\nu}\psi\,. \tag{3.9}$$

We parametrise the two types of fermionic higher derivative terms as

$$\mathcal{L}_f^{\text{HD}} = f_1 i\overline{\psi}\not{\mathcal{D}}\not{\mathcal{D}}\not{\mathcal{D}}\psi + f_2 i\overline{\psi}\not{\mathcal{D}}\mathcal{D}^2\psi + f_3 i\overline{\psi}\mathcal{D}^2\not{\mathcal{D}}\psi + f_4 i\overline{\psi}\mathcal{D}_\mu\not{\mathcal{D}}\mathcal{D}^\mu\psi \tag{3.10}$$

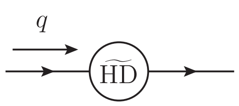
$$\text{and} \quad \mathcal{L}_f^{\widetilde{\text{HD}}} = \tilde{f}_1\overline{\psi}\mathcal{D}^2\psi \quad + \tilde{f}_2 g\, i\,\overline{\psi}F_{\mu\nu}\gamma^{\mu\nu}\psi\,. \tag{3.11}$$

Their contributions to the two-point function yield



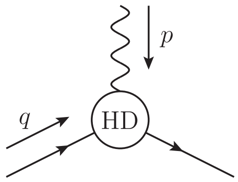
$$= -i(f_1 + f_2 + f_3 + f_4)\not{q}\not{q}\not{q} \tag{3.12}$$

and



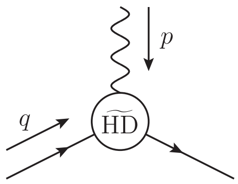
$$= -i\tilde{f}_1 q^2 \tag{3.13}$$

The fermion–gluon vertex for the operators with three space-time derivatives is



$$\begin{aligned} = -igt^a \Big[& f_1\{2(\not{q}+\not{p})q^\mu + \not{q}\not{p}\gamma^\mu + \gamma^\mu(q^2+p^2)\} \\ & + f_2\{2(\not{q}+\not{p})q^\mu + (\not{q}+\not{p})p^\mu + \gamma^\mu q^2\} \\ & + f_3\{\not{q}(2q^\mu + p^\mu) + \gamma^\mu (q+p)^2)\} \\ & + f_4\{\not{q}(2q^\mu + p^\mu) + \not{p}(q^\mu + p^\mu) + \gamma^\mu q\cdot(q+p))\}\Big] \end{aligned} \tag{3.14}$$

and for the operators with two space-time derivatives



$$= igt^a\left(-\tilde{f}_1(2q^\mu + p^\mu) + \tilde{f}_2(\not{p}\gamma^\mu - p^\mu)\right). \tag{3.15}$$

3.2.3. Four Derivative Operators for Gauge Fields

The generic operator with two additional derivatives for the gauge field is

$$\mathrm{tr}\left[F_{\mu\nu}\mathcal{D}_\alpha\mathcal{D}_\beta F_{\rho\sigma}\right] .$$

Due to the anti-symmetry of the fieldstrength tensor, we have three possibilities to contract the index:

$$\mathrm{tr}\left[F^{\mu\nu}\mathcal{D}^\alpha\mathcal{D}_\alpha F_{\mu\nu}\right] , \quad \mathrm{tr}\left[F_{\mu\nu}\mathcal{D}^\mu\mathcal{D}^\rho F_\rho{}^\nu\right] \quad \mathrm{tr}\left[F_{\mu\nu}\mathcal{D}^\rho\mathcal{D}^\mu F_\rho{}^\nu\right]$$

The first of these operators can be expressed as a sum of the latter two using the Bianchi identity

$$\begin{aligned} 0 &= \mathcal{D}_\alpha F_{\mu\nu} + \mathcal{D}_\mu F_{\nu\alpha} + \mathcal{D}_\nu F_{\alpha\mu} \\ \Rightarrow F^{\mu\nu}\mathcal{D}^\alpha\mathcal{D}_\alpha F_{\mu\nu} &= -F^{\mu\nu}\mathcal{D}^\alpha\mathcal{D}_\mu F_{\nu\alpha} - F^{\mu\nu}\mathcal{D}^\alpha\mathcal{D}_\nu F_{\alpha\mu} \end{aligned}$$

Thus, only two terms are needed to capture the higher derivative structure for the gauge fields. We further use the definition of the fieldstrength tensor and the cyclic invariance of the trace. Finally after integrating by parts, we obtain the basis

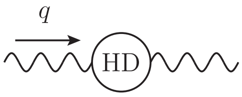
$$\mathrm{tr}\left[\mathcal{D}^\mu F_{\mu\rho}\mathcal{D}_\nu F^{\nu\rho}\right] \quad \text{and} \quad ig\,\mathrm{tr}\left[F^\alpha{}_\beta F^\beta{}_\gamma F^\gamma{}_\alpha\right] \tag{3.16}$$

which has turned out to be the most practical for our calculations. Note that the second term vanishes for an Abelian gauge group.

The gauge boson higher derivatives

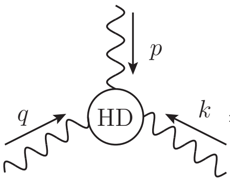
$$\mathcal{L}_{\mathrm{YM}}^{\mathrm{HD}} = v_1\,\mathrm{tr}\left[\mathcal{D}^\mu F_{\mu\rho}\mathcal{D}_\nu F^{\nu\rho}\right] + iv_2 g\,\mathrm{tr}\left[F^\alpha{}_\beta F^\beta{}_\gamma F^\gamma{}_\alpha\right] \tag{3.17}$$

contribute



$$= v_1 i\delta^{ab} q^2 (q^2 \eta^{\mu\nu} - q^\mu q^\nu) \tag{3.18}$$

to the two-point function and



$$\begin{aligned} = g f^{abc} \Big[&\eta^{\mu\nu} \big((p^\rho (2v_1 p \cdot q + v_1 p \cdot k + (3v_1 - \tfrac{3}{2} v_2) q \cdot k) \\ &\qquad - q^\rho (2v_1 q \cdot p + v_1 q \cdot k + (v_1 - \tfrac{3}{2} v_2) p \cdot k) \big) \\ &+ \eta^{\nu\rho} \big((q^\mu (2v_1 q \cdot k + v_1 q \cdot p + (3v_1 - \tfrac{3}{2} v_2) k \cdot p) \big) \\ &\qquad - k^\mu (2v_1 k \cdot q + v_1 k \cdot p + (v_1 - \tfrac{3}{2} v_2) q \cdot p) \\ &+ \eta^{\rho\mu} \big((k^\nu (2v_1 k \cdot p + v_1 k \cdot q + (3v_1 - \tfrac{3}{2} v_2) p \cdot q) \\ &\qquad - p^\nu (2v_1 p \cdot k + v_1 p \cdot q + (v_1 - \tfrac{3}{2} v_2) k \cdot q) \big) \\ &- v_1 \big(k^\mu k^\nu (p^\rho - q^\rho) + p^\nu p^\rho (q^\mu - k^\mu) + q^\rho q^\mu (k^\nu - p^\nu) \big) \\ &- (3v_1 - \tfrac{3}{2} v_2)(p^\rho q^\mu k^\nu - p^\nu q^\rho k^\mu) \Big] \end{aligned} \tag{3.19}$$

to the three-point function.

4. Regularisation and Renormalisation

In this chapter we present the regularisation methods we developed for the computations of this thesis. The standard methods were not suited for the problems we want to tackle. The most popular dimensional regularisation is ruled out, since it regularises all divergences, except logarithmic ones, to zero. The gravitational contributions to the renormalisation of the couplings arise form power like, e. g., quadratic in four space-time dimensions, divergences of the Feynman diagrams. A naive momentum cut-off on the other hand would break gauge invariance and also introduce ambiguities for the non-leading divergent terms. The extra dimensional set-up introduces an other problem for the UV regularisation: The sum over the Kaluza-Klein modes does not converge and needs to be regularised as well. When regularised, the Kaluza-Klein mass and the momentum of the graviton propagator have be treated as one single object, since they are only the projections of the full bulk momentum perpendicular and parallel to the brane. It is also desirable to have unique regulator scale for both, the d-dimensional momenta of the brane fields and the D-dimensional momentum of the graviton. A cut-off regularisation which has this property is introduced in sections 4.1 and 4.2.

Still, gauge invariance is not preserved by the cut-off regulator. In section 4.3, we develop a pre-regularisation scheme which allows us to maintain the gauge invariance of the one-loop amplitudes. The requirement of gauge invariance fixes the parametrisation of the loop momentum, thereby also fixes the ambiguities for the non-leading divergences.

We renormalise the UV divergences of the amplitudes by introducing gauge invariant counterterm into the one-loop effective action. In section 4.4, we show how these counterterms will lead to scale dependent coupling constants and mass terms. We further define the β functions which encode the running of the couplings.

The last section of this chapter contains a small example of the implementation of the calculations in the computer algebra system Form [46].

4.1. Integral Basis

The renormalisation of the parameters of the effective field theory is driven by the necessity to regularise the UV divergences of the one-loop amplitudes. The UV divergences arise from the region of the integration domain where the value of the loop momentum k_μ is larger than the *typical scales* of the amplitudes. The *typical scales* of the amplitudes are the particle masses m and the external momenta, which we will denote by q_μ and p_μ in this chapter. Since we are only interested in the UV divergent and not the exact finite part of the amplitudes, we can expand the one-loop integrals for large k_μ or equivalently for small

q_μ and mass m^2:

$$\begin{aligned}\frac{1}{(k+q)^2-m^2} =& \frac{1}{k^2} - 2\frac{q\cdot k}{k^4} - \frac{q^2-m^2}{k^4} + 4\frac{(q\cdot k)^2}{k^6} + 4\frac{(q^2-m^2)q\cdot k}{k^6} - 8\frac{(q\cdot k)^3}{k^8}\\ &+ \frac{(q^2-m^2)^2}{k^6} - 12\frac{(q^2-m^2)(q\cdot k)^2}{k^8} + 16\frac{(q\cdot k)^4}{k^{10}} + \cdots\end{aligned}$$

The expansion in the mass m^2 is of course only done for the scalar and fermion masses m_ϕ^2 and m_ψ^2. The dependence on the masses $m_{(\vec{n})}^2$ of the Kaluza-Klein gravitons is left unaltered.

After the expansion, the denominators only depend on the absolute value of k and we can exploit the spherical symmetry of the integrand to simplify powers of k in the numerator:

$$\begin{aligned}k^{\text{odd }\#} \to& 0\\ k^\mu k^\nu \to& \tfrac{1}{d}\eta^{\mu\nu}k^2\\ k^\mu k^\nu k^\rho k^\sigma \to& \tfrac{1}{d(d+2)}\underbrace{\left(\eta^{\mu\nu}\eta^{\rho\sigma}+\eta^{\mu\rho}\eta^{\nu\sigma}+\eta^{\mu\sigma}\eta^{\rho\nu}\right)}_{=C^{\mu\nu\rho\sigma}}k^4\\ k^\mu k^\nu k^\rho k^\sigma k^\alpha k^\beta \to& \tfrac{1}{d(d+2)(d+4)}\Big(\eta^{\mu\nu}C^{\rho\sigma\alpha\beta}+\eta^{\mu\rho}C^{\nu\sigma\alpha\beta}\\ &+\eta^{\mu\sigma}C^{\rho\nu\alpha\beta}+\eta^{\mu\alpha}C^{\rho\sigma\nu\beta}+\eta^{\mu\beta}C^{\rho\sigma\alpha\nu}\Big)k^6\,.\end{aligned}$$

For example, the expansion of a bubble integral with up to two powers of the loop momentum in the numerator yields

$$\begin{aligned}\sum_{\vec{n}}\int\frac{\mathrm{d}^dk}{(2\pi)^d}&\frac{Ak^\mu k^\nu + C^\mu k^\nu + C^{\mu\nu}}{((k+p)^2-m_{(\vec{n})}^2)((k+q)^2-m^2)} =\\ &+\tfrac{1}{d}\eta^{\mu\nu}A\sum_{\vec{n}}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{1}{k^2-m_{(\vec{n})}^2} + \Big(\tfrac{1}{d(d+2)}(8q^\mu q^\nu\\ &-(d-2)(q^2-m^2)\eta^{\mu\nu}))A - \tfrac{2}{d}q^\nu C^\mu + C^{\mu\nu}\Big)\sum_{\vec{n}}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{1}{k^2(k^2-m_{(\vec{n})}^2)}\\ &+\Big(\tfrac{1}{d(d+2)}(8p^\mu p^\nu + 4(p^\mu q^\nu + p^\nu q^\mu) + (4q\cdot p - (d-2)p^2)\eta^{\mu\nu}))A\\ &-\tfrac{2}{d}p^\nu C^\mu\Big)\sum_{\vec{n}}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{1}{(k^2-m_{(\vec{n})}^2)^2}\\ &+\mathcal{O}\left(\sum_{\vec{n}}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{1}{k^4(k^2-m_{(\vec{n})}^2)}\right)\,.\end{aligned}\tag{4.1}$$

The resulting basis integrals are of the form:

$$\sum_{\vec{n}\in\mathbb{Z}^\delta}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{m_{(\vec{n})}^{2\alpha}}{k^{2\beta}\left(k^2-m_{(\vec{n})}^2\right)^\gamma}\qquad \alpha,\beta,\gamma\in\mathbb{Z}\,.\tag{4.2}$$

4.2. A common Regulator for Momenta and Kaluza-Klein modes

For the renormalisation we only need the leading (and next-to-leading) UV behaviour of the integrals. So, we can change back from the Kaluza-Klein sums to momentum integrals

$$\sum_{\vec{n}\in\mathbb{Z}^\delta} \quad\longrightarrow\quad \int R^\delta \mathrm{d}^\delta m\,. \tag{4.3}$$

For the sake of clarity we keep the notation and call the compactified momentum m_i. The error introduced by this simplification is suppressed by the inverse of the compactification radius, e. g. the scale M_V from section 2.4, and is less UV divergent. It can be neglected in the spirit of our energy expansion.

Now, one can combine both integrals, the d- and the δ-dimensional, and apply a common cut-off. The combined D-dimensional loop momentum will be named K in the following.

For the tadpole integral this is easily done:

$$\int R^\delta \mathrm{d}^\delta m \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{(k^2-m^2)} = \int \frac{\mathrm{d}^D K}{(2\pi)^d}\frac{R^\delta}{K^2}\,. \tag{4.4}$$

However, most of the integrands we encounter will not only depend on the combination k^2-m^2, but are more general functions of k_μ and m_i. The first step to solve this problem is the integral expansion from section 4.1. The basis integrals have the form

$$\int R^\delta \mathrm{d}^\delta m \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{k^{2\alpha}m^{2\beta}}{k^2-m^2} \tag{4.5}$$

with integers α and β.

In order to eliminate the additional factors of k^2 let us discard the integral over the Kaluza-Klein modes for a moment and have a look at d-dimensional integrals separately. The integrands of the basis integrals only depend on the absolute square of the loop momentum and the finite angular integration can easily be carried out independently from the actual regularisation. This split strictly makes only sense for the Wick rotated, Euclidean integrals. The Wick rotation:

$$k^2 \to -k_E^2 \quad k^2-m^2 \to -(k_E^2+m^2) = -K_E^2 \quad \mathrm{d}^d k \to i\mathrm{d}^d k_E$$

is well defined for our basis integrals, since the integrand has only poles at zero and $m_{(\vec{n})}$.

To illustrate the method we take a Euclidean, d-dimensional integral with the sufficiently general form of the integrand $k_E^{2\alpha}F(k_E^2,m^2)$.

After integration over the angular degrees of freedom we get

$$\int \frac{\mathrm{d}^d k_E}{(2\pi)^d} k_E^{2\alpha} F(k_E^2,m^2) = \frac{\Omega_d}{(2\pi)^d}\int \mathrm{d}k_E\, k_E^{d-1+2\alpha}F(k_E^2,m^2) \tag{4.6}$$

where Ω_d is the volume of the $(d-1)$-dimensional unit sphere S^{d-1}, i. e.,

$$\Omega_d = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})}\,. \tag{4.7}$$

The resulting integral over the radial momentum component could also have arisen from an integral of $F(k_E^2, m^2)$ in $d + 2\alpha$ dimensions. Only the factor from the angular integral differs, but these are know and can be easily corrected by hand. Thus, we have:

$$\begin{aligned}\int \frac{\mathrm{d}^d k_E}{(2\pi)^d} k_E^{2\alpha} F(k_E^2, m^2) &= \frac{\Omega_d}{\Omega_{d+2\alpha}} (2\pi)^{2\alpha} \int \frac{\mathrm{d}^{d+2\alpha} k_E}{(2\pi)^{d+2\alpha}} F(k_E^2, m^2) \\ &= \frac{(4\pi)^\alpha \Gamma(\frac{d}{2}+\alpha)}{\Gamma(\frac{d}{2})} \int \frac{\mathrm{d}^{d+2\alpha} k_E}{(2\pi)^{d+2\alpha}} F(k_E^2, m^2)\,. \end{aligned} \tag{4.8}$$

Note, that we also change the power of the $1/(2\pi)$ factor from the Fourier transformation. This is done purely for convenience. Analogously, we can absorb factors of m into the integration over the Kaluza-Klein modes.

The general basis integral (4.5) can be written as

$$\begin{aligned}\int R^\delta \mathrm{d}^\delta m \int \frac{\mathrm{d}^d k_E}{(2\pi)^d} \frac{k_E^{2\alpha} m^{2\beta}}{(k_E^2 + m^2)^\gamma} &= \frac{(4\pi)^\alpha \Gamma(\frac{d}{2}+\alpha)\Gamma(\frac{\delta}{2}+\beta)}{\pi^\beta \Gamma(\frac{d}{2})\Gamma(\frac{\delta}{2})} \times \\ &\times \int R^\delta \mathrm{d}^{\delta+2\beta} m \int \frac{\mathrm{d}^{d+2\alpha} k_E}{(2\pi)^{d+2\alpha}} \frac{1}{(k_E^2+m^2)^\gamma}\,. \end{aligned} \tag{4.9}$$

Since the integrand now depends only on the combination $k_E^2 + m^2$, we can combine the momentum and mass integration

$$\int R^\delta \mathrm{d}^\delta m \int \frac{\mathrm{d}^d k_E}{(2\pi)^d} \frac{k_E^{2\alpha} m^{2\beta}}{(k_E^2 + m^2)^\gamma} = \frac{(4\pi)^\alpha \Gamma(\frac{d}{2}+\alpha)\Gamma(\frac{\delta}{2}+\beta)}{\pi^\beta \Gamma(\frac{d}{2})\Gamma(\frac{\delta}{2})} \int \frac{\mathrm{d}^{D+2\beta+2\alpha} K_E}{(2\pi)^{d+2\alpha}} \frac{R^\delta}{K_E^{2\gamma}} \tag{4.10}$$

as we did for the tadpole integral.

The final step is the actual regularisation of the integrals which is done by restricting the integration domain of the Euclidean D momentum K_E. The perturbative quantum gravity is only defined as an effective field theory. Thus, it must have an intrinsic theory cut-off we will call Λ. The value of this cut-off is not given a priori, but it is clearly of the order of the smallest of the characteristic scales of the effective field theory we discussed in section 2.4, i. e., the bulk Planck mass $M_{(D)}$ or the mass scale of the brane tension M_τ. From the effective field theory point of view, expressions of the order Λ^n, $n \geq 1$ or proportional to $\log \Lambda$ are effectively UV divergent: Including these in the effective action would lead to a break-down of the energy expansion. In order to maintain a reliable theory at one-loop level, we split the integration domain $0 \leq |K_E| \leq \Lambda$ at an energy μ, the renormalisation or sliding scale [11, 54]. This scale should be of the order of the *typical* scales of the process, e. g., external momenta. The low momentum part $0 \leq |K_E| \leq \mu$ is benign in the energy expansion[1]: It contributes at most $\sim \mu^n$ which will in combination with the dimensionfull coupling constant only yield small numbers as is required for the perturbation theory to be applicable.

The high momentum part $\mu \leq |K_E| \leq \Lambda$ is effectively UV divergent and has to be cancelled by adding suitable counterterms to the one-loop effective action. These counterterms are absorbed into the parameters of the effective theory, i. e., wavefunction normalisation, couplings, mass parameters, and coefficients of the higher derivative operators. Consequently, the renormalised parameters carry a dependence on the scale μ. The renormalisa-

[1]Note, that we neglected all possible IR divergent terms in our integral expansion section 4.1. In the computation of *full* amplitudes these have to be regularised as well.

tion scheme allows for a natural interpretation from the Wilsonian renormalisation group point of view[55, 56]: When we take the limit $\mu \to \Lambda$ the counterterms vanish and renormalised couplings will match the bare couplings, $g_{\text{ren.}}(\mu \to \Lambda) = g_0$, thus the bare couplings are the initial conditions of the renormalisation group flow. Clearly, this limit should be understood as formal, since perturbation theory might break down at this scale.

In the actual computations of chapters 5 and 6, only two of the basis integrals will be needed. The first is the one propagator tadpole integral capturing the divergence of degree $D-2$, e. g., a quadratical divergence in $4+0$ dimensions:

$$\begin{aligned}\int R^\delta \mathrm{d}^\delta m \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2-m^2)} &= -i \int \frac{\mathrm{d}^D K_E}{(2\pi)^d} \frac{R^\delta}{K_E^2} \\ ue &= \frac{-2\pi^{D/2} i}{(2\pi)^d \Gamma(\frac{D}{2})} R^\delta \frac{\Lambda^{D-2} - \mu^{D-2}}{D-2} \,. \end{aligned} \tag{4.11}$$

The second integral we need has a divergence of degree $D-4$, e. g., a logarithmical divergence in $4+0$ dimensions, and is regularised to

$$\begin{aligned}\int R^\delta \mathrm{d}^\delta m \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{k^2(k^2-m^2)} &= \frac{i\Omega_d}{\Omega_{d-2}} \int \frac{\mathrm{d}^{D-2} K_E}{(2\pi)^d} \frac{R^\delta}{K_E^2} \\ &= \frac{2\pi^{D/2} i}{(2\pi)^d \Gamma(\frac{D}{2})} \frac{D-2}{d-2} R^\delta \frac{\Lambda^{D-4} - \mu^{D-4}}{D-4} \,. \end{aligned} \tag{4.12}$$

Since all one-loop integrals we encounter come with a factor κ^2 or to be more precise $\kappa_{(d)}^2$, we will combine the typical prefactor

$$\frac{2\pi^{D/2}}{(2\pi)^d \Gamma(\frac{D}{2})} R^\delta \kappa_{(d)}^2 = \frac{16}{(4\pi)^{D/2-1} \Gamma(\frac{D}{2}) M_{(D)}^{D-2}} \tag{4.13}$$

in the regularised expressions.

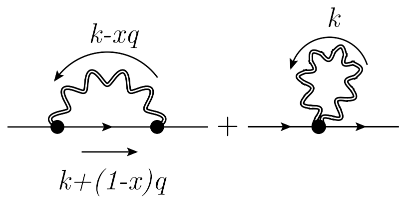
4.3. Parametrisation of the loop momentum

As one can already see in simple examples, in cut-off regularisation the regularised value of the non-leading power-like divergences depends on the parametrisation of the loop momentum. This ambiguity has to be eliminated to extract the structure of higher derivative counterterms in the extra-dimensional scenario[2]. To do so, we demand gauge invariance, with regard to the Yang-Mills gauge group, of the one-loop counterterms and require that all bubbles, triangles, etc. are parametrised in the same manner. This completely fixes the choice of the parametrisation of the loops. To illustrate the procedure we have to anticipate some results from chapter 5. We restrict ourselves to the parts necessary to fix the parametrisation which will be kept explicit. The momentum parametrisation is highly over constrained, but still a solution can be found. We present only single examples for each topology, but all one-loop amplitudes have to yield the same constraints on the parametrisation. This *universality*—at least for all diagrams we encounter in our computations—hints towards an underlying principle that might distinguish the chosen momentum parametrisa-

[2]Without extra dimensions the corresponding divergences are only logarithmic and hence independent of the loop parametrisation.

tion, i. e., the chosen parametrisation does not just work by accident. Actually, one can also motivate the choice of parametrisation we found from theoretical considerations, see below.

The parametrisation of the bubble diagrams can be fixed from the fermion propagator correction. The effective degree of divergence of the bubble graph is $D-1$, in contrast to all further diagrams whose degree of divergence is at most $D-2$. Consequently, also the regularised value of the leading divergence (of degree $D-2$) depends on the chosen loop parametrisation. Using a general distribution of the external momentum q_μ over the two arms of the bubble, the leading divergent term of the one-loop contributions to the two-point function is



$$= \kappa^2 \not{q}\, [\mathcal{C}_1 + \mathcal{C}_2\, x] \sum_{\vec{n}} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{k^2 - m_{\vec{n}}^2} + \cdots \tag{4.14}$$

with

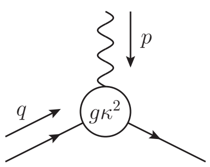
$$\mathcal{C}_1 = \frac{(d-1)\left(8 + 3d(D-4) - 4D + 2d^2(-5+2D)\right)}{32d(D-2)}$$
$$\text{and} \quad \mathcal{C}_2 = \frac{(d-1)(2d-D)}{32D}\,.$$

$0 \le x \le 1$ parametrises the fraction of the external momentum flowing on the graviton line. The requirement of gauge invariance of the counterterms now determines the value of x.

This leading term will contribute to the wavefunction renormalisation of the fermion Z_ψ. As we will discuss in section 6.1, due to the absence of a coupling between Yang-Mills ghosts and gravitons the Slavnov-Taylor identities require the wavefunction and the gauge field vertex renormalisation of a given field to be identical at order κ^2:

$$Z_\psi \Big|_{\mathcal{O}(\kappa^2)} = Z_{\overline{\psi}A\psi} \Big|_{\mathcal{O}(\kappa^2)}\,. \tag{4.15}$$

The effective degree of divergence of the gravitational one-loop diagrams contributing to the fermion–gluon vertex[3] is at most $D-2$ and thus their contribution to $Z_{\overline{\psi}A\psi}$ is independent of the momentum parametrisation:



$$= g\kappa^2 \gamma^\mu \mathcal{C}_3 \sum_{\vec{n}} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{k^2 - m_{\vec{n}}^2} + \cdots \tag{4.16}$$

$$\text{with} \quad \mathcal{C}_3 = \frac{(d-1)\left(8 + 3d(D-4) - 4D + 2d^2(-5+2D)\right)}{32d(D-2)}\,.$$

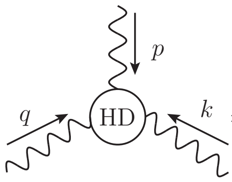
Comparing this result with (4.14) one sees that $\mathcal{C}_1 = \mathcal{C}_3$. Thus, the requirement of the Slavnov-Taylor identities to be satisfied now fixes

$$x = 0\,. \tag{4.17}$$

[3]see Figure 5.5

Once a loop momentum parametrisation is chosen all power-like divergences are fixed. As the regularised integrals will depend on the chosen momentum parametrisation, this particular gauge-compatible parametrisation has to be part of our regularisation scheme. For all graphs with a bubble loop we applied a parametrisation with no external momentum flowing on the graviton line.

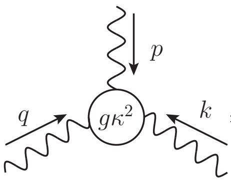
To fix the parametrisation of the triangle diagrams we have a look at the sub-leading divergences (degree $D-4$) of the gluonic three-point amplitude. In order to maintain gauge covariance these have to be absorbed by the higher derivative operators (3.16) in the renormalisation process. If we call the coupling constant of $\operatorname{tr}\left[D^\mu F^2_{\mu\rho}\right]$ v_1 and of $\operatorname{tr}\left[F^3\right]$ v_2 respectively, their contribution to the three point amplitude reads



$$
\begin{aligned}
= g f^{abc}\Big[\eta^{\mu\nu}\big((p^\rho(2v_1 p\cdot q + v_1 p\cdot k + (3v_1 - \tfrac{3}{2}v_2) q\cdot k) - q^\rho(\dots)\big) + \cdots \\
- v_1\big(k^\mu k^\nu(p^\rho - q^\rho) + \cdots\big) \qquad (4.18)\\
- (3v_1 - \tfrac{3}{2}v_2)(p^\rho q^\mu k^\nu - p^\nu q^\rho k^\mu)\Big]\,.
\end{aligned}
$$

The gravitational one-loop contributions to the three-gluon-vertex Figure 5.7 with three external momenta have to be proportional to (4.18). Similarly to the bubble case, we call the fraction of the external momenta on the graviton line y and z, both again between zero and one. Each number is the fraction of the momentum of one of the external lines adjacent to the graviton propagator.

Their relevant part in a general momentum parametrisation is



$$
\begin{aligned}
= g f^{abc}\Big[\eta^{\mu\nu}\big((p^\rho(\mathcal{B}_1 p\cdot q + \mathcal{B}_2 p\cdot k + \mathcal{B}_3 q\cdot k) - q^\rho(\dots)\big) + \cdots \\
+ \mathcal{B}_4\big(k^\mu k^\nu(p^\rho - q^\rho) + \cdots\big) \qquad (4.19)\\
+ \mathcal{B}_5(p^\rho q^\mu k^\nu - p^\nu q^\rho k^\mu)\Big] \sum_{\vec{n}} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{i}{(k^2 - m^2_{\vec{n}})k^2} + \cdots
\end{aligned}
$$

with

$$
\begin{aligned}
\mathcal{B}_1 = \frac{8}{d(d+2)(D-2)}\Big(&2D^4 + D^3(2((y+5z)-7) - 3\delta) \\
&-D^2((14y + 58z + (-12 + y + 7z)\delta) - 32) \\
&+D\left(\left(24y + 56z + 2(-4 + y + 15z)\delta - (-4 + y + 3z)\delta^2 + \delta^3\right) - 24\right) \\
&+2\delta(\delta+2)((2y-\delta) - 2) + 4(\delta(3\delta - 2) + 16)z\Big)\,, \\
\mathcal{B}_2 = \frac{4(d-2)(d((2D+\delta)-6) - 8\delta)}{d(d+2)}\,,&
\end{aligned}
$$

$$\begin{aligned}
\mathcal{B}_3 = & \frac{4}{d\left(d^2-4\right)(D-2)}\Big(6D^5+D^4(-15(\delta+4)+8y+4z) \\
& -3D^3((4y(6+\delta)+2z(6+\delta)-3\delta(12+\delta))-56) \\
& +D^2(3(\delta((-10+\delta)\delta-80)+8)+8(9\delta+26)y+4(9\delta+26)z) \\
& +D(-3\left(\delta(\delta+8)\left(\delta^2-12\right)+176\right)+4(\delta+2)(\delta+6)(\delta-4)y \\
& \quad +2(\delta+2)(\delta+6)(\delta-4)z) \\
& +2\delta\left(3(\delta(-4+\delta(6+\delta))-40)-8(\delta+2)^2y-4(\delta+2)^2z\right)+384\Big) \\
\mathcal{B}_4 = & \frac{4}{d\left(d^2-4\right)(D-2)}\Big(2(4\delta+1)D^5-2D^6-D^4((4y+20z+3\delta(5+4\delta))-28) \\
& +D^3((6y(6+\delta)+2z(78+17\delta)+\delta(-56+\delta(33+8\delta)))-72) \\
& -D^2((4y(26+9\delta)+4z(86+\delta(51+2\delta))+\delta(-100+\delta(-20+\delta(29+2\delta))))-88) \\
& +D((-2y(-4+\delta)(2+\delta)(6+\delta)+z(96+2\delta(124-3(-4+\delta)\delta)) \\
& +\delta(-32+\delta(-28+\delta(16+9\delta))))-48) \\
& +8\left(\delta\left(\left(y(2+\delta)^2-\delta\left(3+\delta^2\right)\right)-2\right)+(\delta+2)(\delta(3\delta-2)+16)z\right)\Big)\,, \\
\mathcal{B}_5 = & -\frac{12}{d\left(d^2-4\right)(D-2)}\Big(\delta^2(d(d((3d+6y)-22)-28y+20)+32(y+3)) \\
& +2(d-2)\left(d\left(28+12d-8d^2+d^3+2(-4+d)(-3+d)y\right)-32\right) \\
& +\delta(d(d(d((5d+10y)-44)-72y+88)+56(3y+2))-128(y+2))\Big)\,.
\end{aligned}$$

Comparison of (4.18) and (4.19) yields the conditions

$$\begin{aligned} 2\mathcal{B}_1 = \mathcal{B}_2 &= -\mathcal{B}_4 \\ \mathcal{B}_3 &= -\mathcal{B}_5 \end{aligned} \tag{4.20}$$

on the parametrisation. The solution of (4.20) is not as obvious as it was for the bubble graph, but the reader might trust or verify herself/himself that the only solution is

$$y = z = 0\,, \tag{4.21}$$

i. e., again no external momentum is allowed on the graviton propagator. The universality of the regularisation scheme—one rule for all triangle graphs—and the demand that the result can be expressed as a linear combination of the gauge invariant operators fixes this unique parametrisation of the loop momentum.

Now we should in principle also fix the parametrisation of the box graphs. Luckily, all boxes we will encounter have a superficial degree of divergence of at most $D-4$, thus we are only interested in their leading UV divergent part. The latter is independent of the momentum parametrisation so it needs not to be fixed.

In the calculation of the renormalisation of the Yukawa and φ^4 interaction where no symmetry principles restrict the structure of the results, we choose the same parametrisation for reasons of consistency.

An alternative motivation for the parametrisation

So far, we presented an empirical approach to the adequate momentum parametrisation. It worked for all diagrams in our calculations and it is the only parametrisation working for all diagrams. But this *ex juvantibus* reasoning is quite unsatisfactory from a theorist's point of view. We should ask if there is a underlying principle which leads to the finding that the momentum of the graviton propagator has to correspond exactly to the loop momentum K. This particular parametrisation can also be motivated from the following consideration. We will not give a complete proof, so the above calculations remain the most important argument for our choice of parametrisation.

We introduced the momentum cut-off in the—for most perturbative calculations—usual way: We wrote down separately each Feynman diagram in momentum space. When we encountered a divergent loop integral, we applied a regularisation prescription. Although the prescription is the same for all diagrams, they are regularised one by one and no symmetry principles restrict the form of the individual diagram. Only the complete amplitude, at each order of perturbation theory, has to obey such principles, e. g., gauge covariance. To see how the parametrisation we choose is connected with gauge covariance, one should compare it with a regularisation scheme which introduces the cut-off commonly for all diagrams. An elegant way to do so is by adding cut-off operators to the action as is done in functional regularisation methods. The cut-off operators would modify the two-point functions of the field in such a way that the propagators are suppressed outside the range of integration.

It is easy to figure out what kind of cut-off operator would lead to a regularisation with similar properties as ours. When we demand that the loop momentum is the momentum of the graviton propagator, it means nothing but that the graviton momentum is restricted to the interval between the cut-offs μ and Λ. So a cut-off operator for the graviton field should be introduced to have the desired effect. The loop propagators of the other fields underlie no explicit restriction. The allowed momentum values stem from the momentum conservation at the vertices. No further cut-off operators are needed.[4] So, only the gravity part of the theory is actually regularised. At tree level, gravity is blind to the internal gauge symmetry. The same is consequently true for the hypothetical functional cut-off operator. Now, it is clear why the parametrisation we choose had the desired property. The regulator breaks diffeomorphism invariance like any momentum cut-off, but leaves the gauge covariance intact.

The above consideration teaches us another lesson: At higher loop order, our strategy work only for the one-loop integral and breaks down at higher loops. Two problems arise when we try to continue our scheme to the next loop level: First, we only regularise the gravity part of the theory. In consequence, to calculate pure gauge, pure Yukawa etc. one-loop running of the couplings, we would need to introduce an additional regulator. Starting at two loops, the gravitational and non-gravitational contributions to the running couplings are entangled, so we could otherwise not determine the complete flow equations.

Second, we would need the renormalisation of the gravitational coupling itself, so we would have to compute purely gravitational amplitudes. For those contributions, e. g., the the one-loop correction to the two-point function of the graviton, our regularisation prescription is nothing but a simple momentum cut-off which breaks diffeomorphism invariance.

[4]Note that we only calculate amplitudes involving one internal graviton propagator.

4.4. Renormalisation

The techniques of the previous sections are used to regularise the UV divergences of the one-loop amplitudes. The original divergences are quantified in terms of the regulators Λ and μ. We can now renormalise the theory by subtraction counterterms proportional to the leading (degree $D-2$) and sub-leading (degree $D-4$) divergences from the bare action. All further divergences are suppressed in terms of the energy expansion. In $D \leq 4$ dimensions no further UV divergences exist at one-loop. As we include only the regularised divergences in the counterterms, our approach is a minimal subtraction scheme.

The counterterms corresponding to the higher derivative operators will stay as they are. We have no information about the values of their associated coupling constants, so an absorption of the counterterms would not yield any new information.

The counterterms proportional to the operators which were present in the original Lagrangian, i. e., kinetic, mass, and interaction terms, will be absorbed and thereby renormalise the bare values of wavefunction normalisation, masses and coupling constants, respectively. To clarify the definitions, we use the φ^4 theory

$$\mathcal{L} = \tfrac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \tfrac{1}{2}m_\varphi^2\varphi^2 - \frac{\lambda}{4!}\varphi^4$$

as an example. The counterterms to this theory are

$$\mathcal{L}_{\text{c.t.}} = \tfrac{1}{2}\delta_\varphi\partial^\mu\varphi\partial_\mu\varphi - \tfrac{1}{2}\delta_{\varphi^2}\varphi^2 - \frac{\lambda}{4!}\delta_{\varphi^4}\varphi^4 \,.$$

We call the counterterms of the form of the kinetic terms δ_{field}. They contribute to the wavefunction renormalisation factors

$$Z_{\text{field}} = 1 + \delta_{\text{field}} \,. \tag{4.22}$$

These Z factors are absorbed into the field variables, e. g.,

$$\varphi_{\text{ren.}} \to Z_\varphi^{-1/2}\varphi_0 \,. \tag{4.23}$$

When we calculate the mass renormalisation for the scalar and Dirac fermion, we have to take into account that part of the divergences δ_{ϕ^2} and $\delta_{\overline{\psi}\psi}$ are already cancelled by the wavefunction Z factors. Only the remainders

$$m_\phi^2\delta_{m_\phi^2} = \delta_{\phi^2} - m_\phi^2\delta_\phi \tag{4.24}$$

$$\text{and} \quad m_\psi\delta_{m_\psi} = \delta_{\overline{\psi}\psi} - m_\psi\delta_\psi \tag{4.25}$$

actually contribute to the mass renormalisation.

Finally, the renormalisation factors for the interaction terms $Z_{\overline{\psi}A\psi} = 1 + \delta_{\overline{\psi}A\psi}$, $Z_{\varphi^4} = 1 + \delta_{\varphi^4}$ etc. are used to compute the β functions

$$\beta_g = \mu\frac{\mathrm{d}}{\mathrm{d}\mu}g \qquad \text{etc.} \tag{4.26}$$

Up to higher loop corrections the bare Lagrangian, e. g.,

$$\mathcal{L}_0 = \tfrac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \tfrac{1}{2}m_{\varphi\,0}^2\varphi^2 - \frac{\lambda_0}{4!}\varphi^4$$

is identical to the renormalised theory $\mathcal{L}+\mathcal{L}_{\text{c.t.}}$ including the Z factors. By comparison, one finds the following relations between the bare and renormalised gauge coupling g, quartic scalar coupling λ, and the Yukawa coupling $\mathcal{Y}$:

$$g_0 = g_{\text{ren.}}\frac{Z_{A^3}}{Z_A{}^{\frac{3}{2}}} \tag{4.27}$$

$$\lambda_0 = \lambda_{\text{ren.}}\frac{Z_{\varphi^4}}{Z_\varphi^2} \tag{4.28}$$

$$\mathcal{Y}_0 = \mathcal{Y}_{\text{ren.}}\frac{Z_{\overline{\psi}\varphi\psi}}{Z_\psi Z_\varphi{}^{\frac{1}{2}}} \tag{4.29}$$

$$\tag{4.30}$$

The bare couplings are naturally independent of the renormalisation process. Thus, by logarithmic differentiation with respect to the renormalisation scale μ and perturbative expansion, i. e., dropping all terms more than linear in the counterterms, we get

$$\beta_g = g\frac{\partial}{\partial\log\mu}\left(\tfrac{3}{2}\delta_A - \delta_{A^3}\right) \tag{4.31}$$

$$\beta_\lambda = \lambda\frac{\partial}{\partial\log\mu}\left(2\delta_\varphi - \delta_{\varphi^4}\right) \tag{4.32}$$

$$\beta_{\mathcal{Y}} = \mathcal{Y}\frac{\partial}{\partial\log\mu}\left(\delta_\psi + \tfrac{1}{2}\delta_\varphi - \delta_{\overline{\psi}\varphi\psi}\right) . \tag{4.33}$$

Before we end our discussion of the renormalisation procedure, a note on the conventions in the following chapters: We compute only the gravitational contributions to the renormalisation and will not denote the gravitational nature in the results, i. e., when we write e. g., δ_ψ we mean only the gravitational part of the fermion wavefunction renormalisation and not the full one loop corrections. Only in case we want to clarify special features of their gravitational part, we will explicitly tag the corresponding quantities, e. g., $\delta_\psi\big|_{\mathcal{O}(\kappa^2)}$.

4.5. Implementation

In chapter 2 and the first sections of this chapters, we explained the steps of our calculations. Most of the computations are hardly feasible by pen and paper. Therefore, we used Jos Vermaseren's computer algebra system Form [46]. Form's scripting language allows for are direct computer implementation of the term manipulations one usually does by hand without the need of adapting our established algorithms from perturbative quantum field theory calculations. Only some unhandy algebra, like the tests of the loop parametrisation section 4.3, was done using Mathematica by Wolfram Research Inc.

Let us illustrate the Form implementation of the algorithm by means of the gravitational one-loop contributions to the two-point function of the scalar field[5]. We start by defining

[5]The complete, uninterrupted example script can be found in Appendix B.

the variables:

```
*gravitational coupling expansion parameter with maximum power 2
symbol ka(:2);

*scalar mass, brane dimension & additional symbols for internal use
symbol MassS,D,[D+2],[D+4],[De-2],[de-2],de,De;

*fraction of the external momentum on the graviton line
symbol x;

*square root of metric determinat, inverse metric, graviton
tensors sqrtg,gI(s),h(s),H;

*momenta (external, loop & additional tensors for internal use)
tensors p,q,K,p1,p2,[-q],[-K],[K+p2],[-K-p2];

*function for symmetrisation
function Sym,P;

*indices (fixed)
dimension 0;
autodeclare index a;

*indices (summable)
dimension D;
indices mu,nu,a,b,c,d;
autodeclare index i;

*vector versions of the momenta
vectors qV,p1V,p2V;
```

Now, we declare an expression containing the Lagrangian quadratic in the scalar field. We already preformed twice the functional derivative with respect to the scalar field:

```
Local Lphi2=sqrtg*(gI(mu,nu)*(-i_)*p(mu)*(-i_)*q(nu) -MassS^2);
```

`p` and `q` are the ingoing momenta of the scalar.

We then substitute the metric dependent quantities by their expansion in the graviton field `h`, taking special care of the dummy indices:

```
repeat;
id,once,sqrtg=1+1/2*ka*h(i1,i1)
               +1/8*ka*ka*(h(i1,i1)*h(i2,i2)-2*h(i1,i2)*h(i1,i2));
id,once,gI(mu?,nu?)=d_(mu,nu)-ka*h(mu,nu)+ka*ka*h(mu,i1)*h(nu,i1);
sum i1,i2;
endrepeat;

.sort
```

and substitute the indices of the graviton by fixed ones, i. e., indices of dimension zero. We use a new four-index function for two gravitons:

```
id h(i1?,i2?)*h(i3?,i4?)=H(a1,a2,a3,a4)
                           *(d_(a1,i1)*d_(a2,i2)*d_(a3,i3)*d_(a4,i4)
                            +d_(a1,i3)*d_(a2,i4)*d_(a3,i1)*d_(a4,i2)
                            +d_(a2,i1)*d_(a1,i2)*d_(a3,i3)*d_(a4,i4)
                            +d_(a2,i3)*d_(a1,i4)*d_(a3,i1)*d_(a4,i2)
                            +d_(a1,i1)*d_(a2,i2)*d_(a4,i3)*d_(a3,i4)
                            +d_(a1,i3)*d_(a2,i4)*d_(a4,i1)*d_(a3,i2)
                            +d_(a2,i1)*d_(a1,i2)*d_(a4,i3)*d_(a3,i4)
                            +d_(a2,i3)*d_(a1,i4)*d_(a4,i1)*d_(a3,i2))/8;

id h(i1?,i2?)=h(a1,a2)*(d_(a1,i1)*d_(a2,i2)+d_(a2,i1)*d_(a1,i2))/2;
```

We set the expansion parameter to one—it is not needed any more—and factor out the gravitons:

```
id ka=1;

bracket h,H;

.sort
```

In the next module, we define vertex expressions and two functions we will use to substitute the vertices in the diagrams:

```
functions fun1gr2scalar,fun2gr2scalar;

Local V1gr2scalar= i_*Lphi2[h(a1,a2)];
Local V2gr2scalar= i_*2*Lphi2[H(a1,a2,a3,a4)];
```

In all vertex expressions, we have at most one internal index. To make the insertion of the vertex function easier, we name it `i` in all expressions:

```
id p?(i1?!{a1,a2,a3,a4})=p(i);
id d_(i1?!{a1,a2,a3,a4},i2?)=d_(i,i2);

.sort
```

After this prelude, we define expressions for the one-loop diagrams and insert the vertices we obtained above:

```
*propagator correction with 2 graviton-scalar (1+2) vertices (bubble)
Local PC1=-fun1gr2scalar(q,[-K-p2],a,b)*P(a,b,c,d)
                        *fun1gr2scalar([K+p2],[-q],c,d);

*propagator correction with 1 graviton-scalar (2+2) vertex (Seagull)
Local PC2=i_/2*fun2gr2scalar(q,[-q],a,b,c,d)*P(a,b,c,d);

*defining the vertex functions
```

```
repeat;
id,once,fun1gr2scalar(q?,p?,a1?,a2?)=V1gr2scalar;
sum i;
endrepeat;

repeat;
id,once,fun2gr2scalar(q?,p?,a1?,a2?,a3?,a4?)=V2gr2scalar;
sum i;
endrepeat;

*numerator of the graviton propagator
id P(a?,b?,c?,d?)=1/2*(d_(a,c)*d_(b,d)+d_(a,d)*d_(b,c))
                  -d_(a,b)*d_(c,d)/[De-2];
```

Since sums as function arguments can yield some unforeseeable problems in FORM, we first use some place holder tensors, which we now substitute

```
id [-q](i?)=-q(i);

id [-K-p2](i?)=-K(i)-p2(i);
id [K+p2](i?)=K(i)+p2(i);

bracket K;

.sort
```

To be able to use our integral expansion (4.1), we substitute the indices of the loop momentum `K` by fixed indices `a1,a2`:

```
id only K(mu?!{a1,a2})*K(nu?!{a1,a2})=K(a1)*K(a2)*Sym(a1,mu,a2,nu);
id only K(mu?!{a1,a2})=K(a1)*d_(a1,mu);

id Sym(a1?,i1?,a2?,i2?)=(d_(a1,i1)*d_(a2,i2)+d_(a2,i1)*d_(a1,i2))/2;

bracket K;

.sort
```

Now, we can define expressions which are proportional to the basis integrals:

```
L leadPC1=PC1[K(a1)*K(a2)]*Sym(a1,a2);
L leadPC2=PC2[1];

L subleadPC=PC1[1]*(De-2)/[de-2]
         -2*PC1[K(a1)]*(Sym(a1,p1V)+Sym(a1,p2V)*(De-2)/[de-2])
         +PC1[K(a1)*K(a2)]*(4*(Sym(a1,a2,p1V,p1V)*D/De
                              +Sym(a1,a2,p1V,p2V)
                              +Sym(a1,a2,p2V,p2V)*(De-2)/[de-2])
                           -(p1V.p1V
                             +(p2V.p2V-MassS^2)*(De-2)/[de-2])
```

```
                                                    *Sym(a1,a2));

id Sym(a1?,a2?,i1?,i2?)=(d_(a1,a2)*Sym(i1,i2)+d_(a1,i2)*Sym(i1,a2)
                         +d_(a1,i1)*Sym(i2,a2))/[D+2];
id Sym(a1?,a2?)=d_(a1,a2)/D;

.sort
```

In the end, the expressions `lead...` have to be multiplied with the regularised integral (4.11), the expressions `sublead...` have to be multiplied with the regularised integral (4.12), respectively.

The algebra with space-time indices is done now and the indices need no special care any more. Thus, we make the fixed indices sumable and change from momentum tensors to vectors:

```
index a1=D,a2=D,a3=D,a4=D;

*express momenta by a vector instead of a tensor
tovector q  qV;
tovector p1 p1V;
tovector p2 p2V;

.sort
```

Until now, the momenta of the internal lines were `K+p1` and `K+p2`, respectively. We introduce the parameter `x` of the fraction of the external momentum like in section 4.3:

```
if (expression(subleadPC));
id p1V=-qV*x;
id p2V=qV*(1-x);
endif;

.sort
```

Finally, we collect the terms proportional to (4.11) and bring the expressions to a format which can be evaluated by Mathematica:

```
L leadPC=leadPC1+leadPC2;

id D^x?=de^x;
id [D+2]^x?=(de+2)^x;
id [D+4]^x?=(de+4)^x;
id [de-2]^x?=(de-2)^x;
id [De-2]^x?=(De-2)^x;

bracket qV,MassS;

format mathematica;

print leadPC,subleadPC;
```

```
.end
```

Here, de is the brane dimension d and De is the bulk dimension D.

For the test of the loop parametrisation section 4.3, we leave the momentum fraction on the graviton line x arbitrary. After we convinced ourselves that the parametrisation without any external momentum in the graviton propagator is preferable, we can also fix the parametrisation

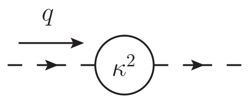
```
id x=0;
```

before printing the results.

The above script (with the fixed parametrisation $x = 0$) will yield the result

```
subleadPC =
     + MassS^4 * (  - 1/2/( - 2 + de)*de + 1/4/( - 2 + de)*de*De + 1/
       2/( - 2 + de)/( - 2 + De)*de^2 - 1/4/( - 2 + de)/( - 2 + De)*
       de^2*De )

     + qV.qV*MassS^2 * (  - 11/2/( - 2 + de) + 5/( - 2 + de)*de^(-1)
        - 5/2/( - 2 + de)*de^(-1)*De + 11/4/( - 2 + de)*De + 1/( - 2
        + de)*de - 1/2/( - 2 + de)*de*De - 6/( - 2 + de)/( - 2 + De)
        + 2/( - 2 + de)/( - 2 + De)*de^(-1) - 1/( - 2 + de)/( - 2 +
       De)*de^(-1)*De + 3/( - 2 + de)/( - 2 + De)*De + 9/2/( - 2 + de
       )/( - 2 + De)*de - 9/4/( - 2 + de)/( - 2 + De)*de*De - 1/( - 2
        + de)/( - 2 + De)*de^2 + 1/2/( - 2 + de)/( - 2 + De)*de^2*De
        )

     + qV.qV^2 * ( 7/2/( - 2 + de) - 1/( - 2 + de)*de^(-1) + 1/2/( -
       2 + de)*de^(-1)*De - 7/4/( - 2 + de)*De - 1/2/( - 2 + de)*de
        + 1/4/( - 2 + de)*de*De + 12/( - 2 + de)/( - 2 + De) - 10/(
        - 2 + de)/( - 2 + De)*de^(-1) + 5/( - 2 + de)/( - 2 + De)*
       de^(-1)*De - 6/( - 2 + de)/( - 2 + De)*De - 9/2/( - 2 + de)/(
        - 2 + De)*de + 9/4/( - 2 + de)/( - 2 + De)*de*De + 1/2/( - 2
        + de)/( - 2 + De)*de^2 - 1/4/( - 2 + de)/( - 2 + De)*de^2*De
        - 10/(2 + de)/( - 2 + de) + 4/(2 + de)/( - 2 + de)*de^(-1) -
       2/(2 + de)/( - 2 + de)*de^(-1)*De + 5/(2 + de)/( - 2 + de)*De
        - 24/(2 + de)/( - 2 + de)/( - 2 + De) + 24/(2 + de)/( - 2 +
       de)/( - 2 + De)*de^(-1) - 12/(2 + de)/( - 2 + de)/( - 2 + De)*
       de^(-1)*De + 12/(2 + de)/( - 2 + de)/( - 2 + De)*De + 6/(2 +
       de)/( - 2 + de)/( - 2 + De)*de - 3/(2 + de)/( - 2 + de)/( - 2
        + De)*de*De );

 leadPC =
     + MassS^2 * (  - 1/8*de^2 + 1/4/( - 2 + De)*de - 1/8/( - 2 + De)
       *de^2 )

     + qV.qV * ( 3/4 - 1/2*de^(-1) - 1/2*de + 1/8*de^2 + 2/( - 2 + De
       ) - 1/( - 2 + De)*de^(-1) - 1/( - 2 + De)*de + 1/8/( - 2 + De)
       *de^2 );
```

The gravitational one-loop contributions to the scalar two-point function (A.1)



$$= \frac{-2\pi^{D/2} i}{(2\pi)^d \Gamma(\frac{D}{2})} R^\delta \frac{\Lambda^{D-2} - \mu^{D-2}}{D-2} \times \texttt{leadPC}$$
$$+ \frac{2\pi^{D/2} i}{(2\pi)^d \Gamma(\frac{D}{2})} \frac{D-2}{d-2} R^\delta \frac{\Lambda^{D-4} - \mu^{D-4}}{D-4} \times \texttt{subleadPC}$$

can be read off these results using the basis integrals (4.11) and (4.12).

5. Wavefunction Renormalisation and Higher Derivative Counterterms

Before we calculate the renormalisation of the Standard Model couplings in the next chapter, we look at the individual fields and determine the gravitational corrections to their wavefunction and mass renormalisation. In addition to these quantities which are solely fixed by propagator corrections, we compute the renormalisation of the gauge covariant higher derivative terms introduced in chapter 3. To do so, we need to calculate also the one-loop contributions to the coupling of the fields to gauge bosons.

As in [53], we present most of the results in this chapter only for the $4+\delta$ dimensional case. An exception are branon results in section 5.1 which are valid for arbitrary brane dimensions. The full results for $d+\delta$ dimensions can be found in Appendix A.

Some remarks on the conventions

The symmetry factors of all one-loop graphs are easily determined, due to their simple structure of exactly one internal graviton line dressing a tree level graph of the other fields. All diagrams with a single loop propagator, i. e., tadpole- and seagull-like diagrams, contribute with a symmetry factor of 1/2. All other diagrams contribute with weight one. A detailed derivation of the symmetry factors was done in [16, app. A].

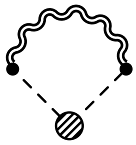
The reader might notice that in all Feynman graphs of this chapter we only draw the wiggly double lines which were used to represent the tensor part of the graviton in chapter 2. Clearly, the full one-loop amplitudes also include the scalar part of the graviton, i. e., the straight double line. As the latter couples to the brane matter fields in the same manner as the first one, except of course in the index structure, all one-loop diagrams could be drawn twice: Once the wiggly and once with straight lines. For the sake of clarity we refrain from drawing the latter and all double lines should be understood as the combined propagator, i. e.,

$$ {}^{(\vec{n})}_{\alpha\beta} \quad {}^{(\vec{n}')}_{\gamma\delta} = \frac{i\delta_{\vec{n},-\vec{n}'}\frac{1}{2}\left(\eta_{\alpha\gamma}\eta_{\beta\delta}+\eta_{\alpha\delta}\eta_{\beta\gamma}-\frac{2}{D-2}\eta_{\alpha\beta}\eta_{\gamma\delta}\right)}{p^2-m_{\vec{n}}^2}\,, \tag{5.1} $$

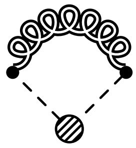
for the tensor *and* the scalar part of the graviton's degrees of freedom.

5.1. Contributions from Branon Coupling

Although we are only interested in pure gravitational one-loop contributions and will not consider any terms involving the brane tension τ, we cannot ignore the branons completely due to the inverse τ dependence in matter–branon interactions and graviton–branon mixing [57], see Figure 2.1 and equation (2.26). There exist two shapes of tadpole graphs involving branons at order κ^2. These were calculated using the general form of the interactions, Figure 2.1, so the results are applicable for generic brane matter fields. In particular, for the parachute shaped graphs we get

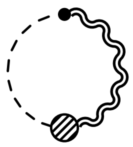


$$= -\frac{\kappa^2}{4} T_\mu{}^\mu \frac{\delta - 2}{D-2} \sum_{\vec{n}} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{m_{\vec{n}}^2}{k^2(k^2 - m_{\vec{n}}^2)}$$

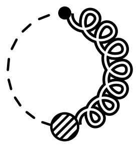


$$= \frac{\kappa^2}{2} T_\mu{}^\mu \frac{\delta}{d} \sum_{\vec{n}} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{k^2 - m_{\vec{n}}^2} \tag{5.2}$$

and for graphs whose shape might be reminiscent of an early crescent we have



$$= \frac{\kappa^2}{4} T_\mu{}^\mu \frac{\delta - 2}{D-2} \sum_{\vec{n}} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{m_{\vec{n}}^2}{k^2(k^2 - m_{\vec{n}}^2)}$$



$$= -\frac{\kappa^2}{2} T_\mu{}^\mu \frac{\delta}{d} \sum_{\vec{n}} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{k^2 - m_{\vec{n}}^2} \,. \tag{5.3}$$

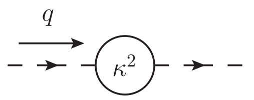
Here $T_\mu{}^\mu$ is the trace of the energy momentum tensor and the sum runs over the Kaluza-Klein gravitons. Clearly, the formal expressions for the momentum integrals in equations (5.2) and (5.3) are understood to be suitably regularised. We did not apply the integral expansion from section 4.1 so the expressions are exact at one loop level.

From the above expressions it directly follows that the sum of all branon tadpole $\sim \kappa^2$ graphs is zero, independently of the number of the compactified dimensions. This means, that the branon effects which independent of the the brane tension and would contribute with the same magnitude as the pure gravitational corrections vanish.

Figure 5.1.: One-loop diagrams for the proper scalar two-point function.

5.2. Counterterms involving the Scalar Field

We start with the self-energy. Its gravitational part is determined by the diagrams in Figure 5.1:



$$= \frac{i}{(4\pi)^{1+\delta/2}\Gamma\left(\frac{\delta}{2}+2\right)}\Bigg[m_\phi^2\left\{\frac{16(5+2\delta)}{(\delta+2)^2}\frac{\Lambda^{\delta+2}-\mu^{\delta+2}}{M_{(D)}^{\delta+2}} + \frac{8(\delta-2)}{\delta}m_\phi^2\frac{\Lambda^{\delta}-\mu^{\delta}}{M_{(D)}^{\delta+2}}\right\} - q^2\left\{\frac{2(8+5\delta)}{(\delta+2)^2}\frac{\Lambda^{\delta+2}-\mu^{\delta+2}}{M_{(D)}^{\delta+2}} - \frac{16+\delta}{\delta}m_\phi^2\frac{\Lambda^{\delta}-\mu^{\delta}}{M_{(D)}^{\delta+2}}\right\} + q^4\frac{\Lambda^{\delta}-\mu^{\delta}}{M_{(D)}^{\delta+2}}\Bigg] + \cdots . \tag{5.4}$$

Note that the higher derivative divergence $\sim q^4$ vanishes if extra-dimensions are absent ($\delta = 0$). Hence, in the standard gravity set-up without extra-dimensions there are no gravitational contributions to the renormalisation the scalar Lee-Wick term $(D^2\phi)^\dagger D^2\phi$.

The one-loop diagrams for the two-scalar–one-gauge-field vertex are listed in Figure 5.2.

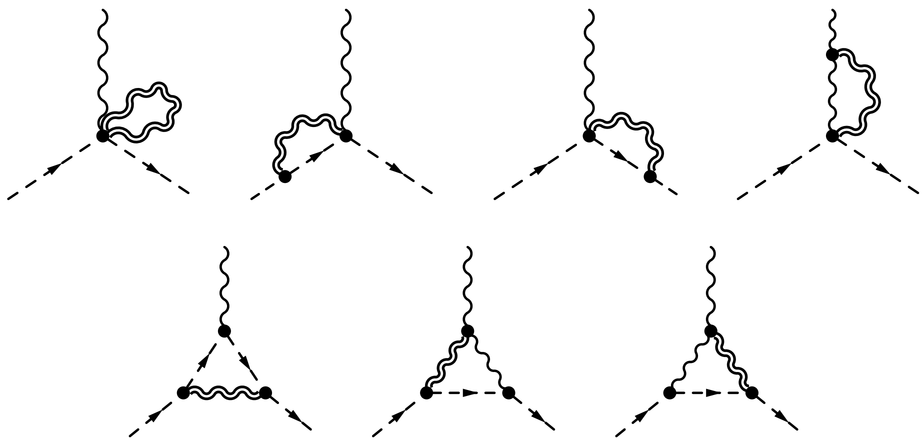
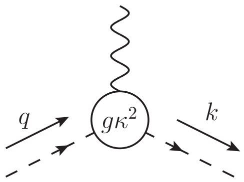


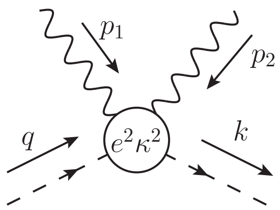
Figure 5.2.: One-loop diagrams for the proper two-scalars–one-gauge-boson vertex.

Their sum is



$$
= \frac{i\,gt^a}{(4\pi)^{1+\delta/2}\Gamma\left(\frac{\delta}{2}+2\right)}\Bigg[-(q+k)^\mu\bigg\{\frac{2(8+5\delta)}{(\delta+2)^2}\frac{\Lambda^{\delta+2}-\mu^{\delta+2}}{M_{(D)}^{\delta+2}} - \frac{16+\delta}{\delta}m_\phi^2\frac{\Lambda^\delta-\mu^\delta}{M_{(D)}^{\delta+2}}\bigg\} + \Big\{(q^2+\tfrac{1}{3}qk+\tfrac{2}{3}k^2)q^\mu + (k^2+\tfrac{1}{3}kq+\tfrac{2}{3}q^2)k^\mu\Big\}\frac{\Lambda^\delta-\mu^\delta}{M_{(D)}^{\delta+2}}\Bigg] + \cdots . \tag{5.5}
$$

Finally, to determine the counterterm for the operator $\phi^\dagger F^{\mu\nu}F_{\mu\nu}\phi$ the two-scalars–two-gauge-bosons amplitude is necessary. To simplify the calculation, it is done for an *Abelian* theory with the Abelian coupling constant e which already leads to the 29 one-loop diagrams listed in Figure 5.3. All these diagrams sum up to



$$
= \frac{i\,e^2}{(4\pi)^{1+\delta/2}\Gamma\left(\frac{\delta}{2}+2\right)}\Bigg[-2\eta^{\mu\nu}\bigg\{\frac{2(8+5\delta)}{(\delta+2)^2}\frac{\Lambda^{\delta+2}-\mu^{\delta+2}}{M_{(D)}^{\delta+2}} - \frac{16+\delta}{\delta}m_\phi^2\frac{\Lambda^\delta-\mu^\delta}{M_{(D)}^{\delta+2}}\bigg\} + \Big\{2(q+k)^\mu(q+k)^\nu + (q+k)^2\eta^{\mu\nu} + \tfrac{1}{3}p_1^\mu p_1^\nu - 2p_1^\nu p_2^\mu + \tfrac{1}{3}p_2^\mu p_2^\nu + \eta^{\mu\nu}(\tfrac{2}{3}p_1^2 + 2p_1p_2 + \tfrac{2}{3}p_2^2)\Big\}\frac{\Lambda^\delta-\mu^\delta}{M_{(D)}^{\delta+2}}\Bigg] + \cdots . \tag{5.6}
$$

The gravitational corrections to the *non*-Abelian scalar–two-gauge-bosons interaction do not differ form the Ablian one, because gravity is insensitive to the particular choice of the gauge group. Since the scalar field higher derivative operators for Abelian and non-Abelian groups have the identical structure, this result can easily be generalized to a non-Abelian theory.

By comparing the divergent terms in (5.4)–(5.6) with the Feynman rules for the higher derivative operators (3.4)–(3.6), we obtain the gravitational counterterms involving two

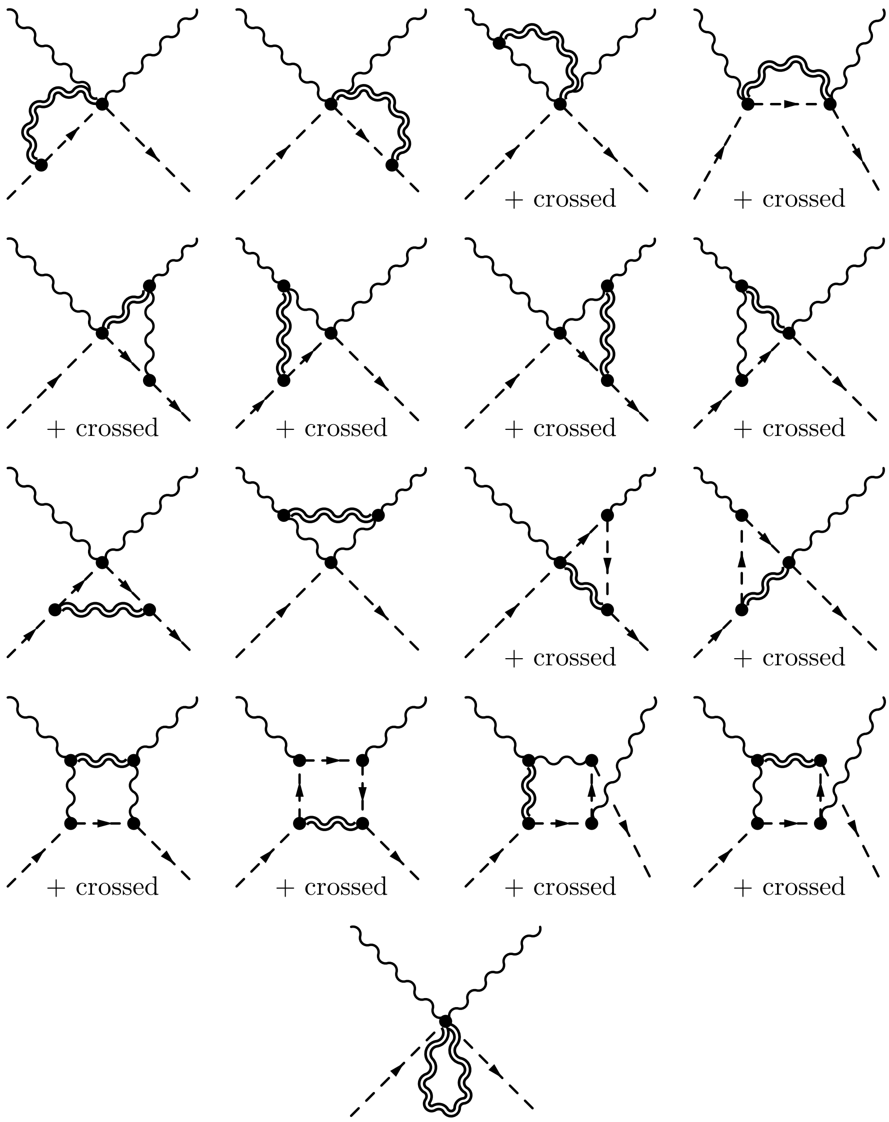


Figure 5.3.: One-loop diagrams for the two-scalars–two-photons vertex. "+ crossed" refers to the corresponding diagram with exchanged external photon lines.

scalars

$$\mathcal{L}_s^{\text{c.t.}} = \frac{1}{(4\pi)^{1+\delta/2}\Gamma\left(\frac{\delta}{2}+2\right)}\Bigg[(\mathcal{D}_\mu\phi)^\dagger\mathcal{D}^\mu\phi\left\{\frac{2(8+5\delta)}{(\delta+2)^2}\frac{\Lambda^{\delta+2}-\mu^{\delta+2}}{M_{(D)}^{\delta+2}} - \frac{16+\delta}{\delta}m_\phi^2\frac{\Lambda^\delta-\mu^\delta}{M_{(D)}^{\delta+2}}\right\}$$
$$- m_\phi^2\left\{\frac{16(5+2\delta)}{(\delta+2)^2}\frac{\Lambda^{\delta+2}-\mu^{\delta+2}}{M_{(D)}^{\delta+2}} + \frac{8(\delta-2)}{\delta}m_\phi^2\frac{\Lambda^\delta-\mu^\delta}{M_{(D)}^{\delta+2}}\right\}$$
$$- \left\{(\mathcal{D}^2\phi)^\dagger\mathcal{D}^2\phi - \tfrac{1}{3}ig(\mathcal{D}_\mu\phi)^\dagger F^{\mu\nu}\mathcal{D}_\nu\phi + \tfrac{1}{6}g^2\phi^\dagger F^{\mu\nu}F_{\mu\nu}\phi\right\}\frac{\Lambda^\delta-\mu^\delta}{M_{(D)}^{\delta+2}}\Bigg]. \quad (5.7)$$

From the counterterms (5.7) we can determine the gravitational contributions to the wavefunction and mass renormalisation of the scalar field

$$\delta_\phi = \frac{1}{(4\pi)^{1+\delta/2}\Gamma\left(\frac{\delta}{2}+2\right)}\left\{\frac{2(8+5\delta)}{(\delta+2)^2}\frac{\Lambda^{\delta+2}-\mu^{\delta+2}}{M_{(D)}^{\delta+2}} - \frac{16+\delta}{\delta}m_\phi^2\frac{\Lambda^\delta-\mu^\delta}{M_{(D)}^{\delta+2}}\right\} \quad (5.8)$$

$$\delta_{m_\phi^2} = \frac{1}{(4\pi)^{1+\delta/2}\Gamma\left(\frac{\delta}{2}+2\right)}\left\{\frac{2(32+11\delta)}{(\delta+2)^2}\frac{\Lambda^{\delta+2}-\mu^{\delta+2}}{M_{(D)}^{\delta+2}} + 9\,m_\phi^2\frac{\Lambda^\delta-\mu^\delta}{M_{(D)}^{\delta+2}}\right\} \quad (5.9)$$

The mass renormalisation was defined in (4.24) as $m_\phi^2\delta_{m_\phi^2} = \delta_{\phi^2} - m_\phi^2\delta_\phi$.

Figure 5.4.: One-loop diagrams for the proper fermion two-point function.

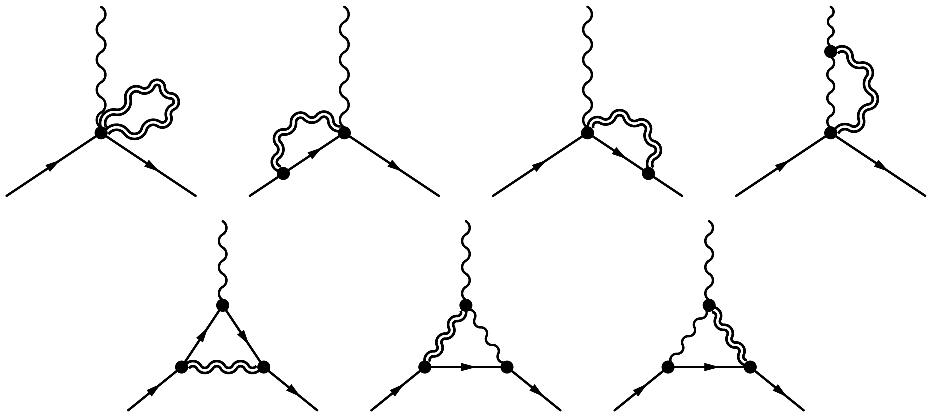
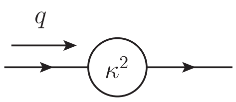


Figure 5.5.: One-loop diagrams for the proper two-fermions–one-gauge-boson vertex.

5.3. Counterterms for Dirac Fermions

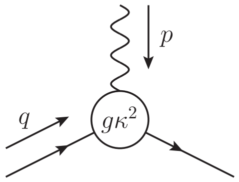
The gravitational one-loop contributions to the two-point function of the Dirac fermion are shown in Figure 5.4.



$$
\begin{aligned}
= \frac{i}{(4\pi)^{1+\delta/2}\Gamma\left(\frac{\delta}{2}+2\right)} \Bigg[m_\psi \Bigg\{ & \frac{79\delta+112}{2(\delta+2)^2} \frac{\Lambda^{\delta+2}-\mu^{\delta+2}}{M_{(D)}^{\delta+2}} \\
& + \frac{17\delta-16}{4\delta} m_\psi^2 \frac{\Lambda^{\delta}-\mu^{\delta}}{M_{(D)}^{\delta+2}} \Bigg\} \\
- \not{q} \Bigg\{ \frac{3(11+9\delta)}{(\delta+2)^2} \frac{\Lambda^{\delta+2}-\mu^{\delta+2}}{M_{(D)}^{\delta+2}} & + \frac{19\delta-32}{8\delta} m_\psi^2 \frac{\Lambda^{\delta}-\mu^{\delta}}{M_{(D)}^{\delta+2}} \Bigg\} \\
+ \not{q}\not{q} m_\psi \frac{3(\delta+4)}{2\delta} \frac{\Lambda^{\delta}-\mu^{\delta}}{M_{(D)}^{\delta+2}} & - \not{q}\not{q}\not{q} \frac{16-\delta}{8\delta} \frac{\Lambda^{\delta}-\mu^{\delta}}{M_{(D)}^{\delta+2}} \Bigg] + \cdots
\end{aligned}
\tag{5.10}
$$

As we already discussed in chapter 3, self-gravitational effects renormalise not only fermionic operators with two additional derivatives, but also those with only one additional derivative.

For the full structure of the renormalisation of the higher derivative terms we need the gravitational corrections to the coupling between the fermion and the gauge field, Figure 5.5:



$$
\begin{aligned}
= \frac{i\,gt^a}{(4\pi)^{1+\delta/2}\Gamma\left(\frac{\delta}{2}+2\right)}\Bigg[& -\gamma^\mu\Bigg\{\frac{3(11+9\delta)}{(\delta+2)^2}\frac{\Lambda^{\delta+2}-\mu^{\delta+2}}{M_{(D)}^{\delta+2}} \\
& \qquad + \frac{19\delta-32}{8\delta}m_\phi^2\frac{\Lambda^\delta-\mu^\delta}{M_{(D)}^{\delta+2}}\Bigg\} \\
& + \{2q^\mu+p^\mu\}m_\psi\frac{3(\delta+4)}{2\delta}\frac{\Lambda^\delta-\mu^\delta}{M_{(D)}^{\delta+2}} \\
& + \{\not{p}\gamma^\mu-p^\mu\}\frac{3(3\delta+4)}{2\delta}\frac{\Lambda^\delta-\mu^\delta}{M_{(D)}^{\delta+2}} \\
& + \Bigg\{-\frac{16-\delta}{4}\not{q}q^\mu+\frac{32+25\delta}{4}\not{q}p^\mu-\frac{80+49\delta}{8}\not{q}\not{p}\gamma^\mu \\
& - 6(2+\delta)\not{p}q^\mu-\frac{17(2+\delta)}{6}\not{p}p^\mu-\frac{16-\delta}{8}\gamma^\mu q^2 \\
& + \frac{32+25\delta}{4}\gamma^\mu qp+\frac{88+71\delta}{24}\gamma^\mu p^2\Bigg\}\frac{\Lambda^\delta-\mu^\delta}{\delta M_{(D)}^{\delta+2}}\Bigg] + \cdots\,.
\end{aligned}
\tag{5.11}
$$

Now, we can determine the counterterms using the Feynman rules (3.12) and (3.14) from chapter 3:

$$
\begin{aligned}
\mathcal{L}_f^{\text{c.t.}} = \frac{1}{(4\pi)^{1+\delta/2}\Gamma\left(\frac{\delta}{2}+2\right)}\overline{\psi}\Bigg[& \Bigg\{\frac{79\delta+112}{2(\delta+2)^2}\frac{\Lambda^{\delta+2}-\mu^{\delta+2}}{M_{(D)}^{\delta+2}}+\frac{17\delta-16}{4\delta}m_\psi^2\frac{\Lambda^\delta-\mu^\delta}{M_{(D)}^{\delta+2}}\Bigg\}m_\psi \\
& i\Bigg\{\frac{3(11+9\delta)}{(\delta+2)^2}\frac{\Lambda^{\delta+2}-\mu^{\delta+2}}{M_{(D)}^{\delta+2}}+\frac{19\delta-32}{8\delta}m_\psi^2\frac{\Lambda^\delta-\mu^\delta}{M_{(D)}^{\delta+2}}\Bigg\}\not{\mathcal{D}} \\
& + \frac{3(\delta+4)}{2\delta}\frac{\Lambda^\delta-\mu^\delta}{M_{(D)}^{\delta+2}}\mathcal{D}^2 \\
& + i\,g\frac{3(3\delta+4)}{2\delta}\frac{\Lambda^\delta-\mu^\delta}{M_{(D)}^{\delta+2}}F_{\mu\nu}\gamma^{\mu\nu} \\
& + i\Big\{-(10+\tfrac{49}{8}\delta)\not{\mathcal{D}}\not{\mathcal{D}}\not{\mathcal{D}}+(\tfrac{41}{3}+\tfrac{109}{12}\delta)\not{\mathcal{D}}\mathcal{D}^2 \\
& + (\tfrac{41}{3}+\tfrac{109}{12}\delta)\mathcal{D}^2\not{\mathcal{D}}-(\tfrac{58}{3}+\tfrac{143}{12}\delta)\mathcal{D}_\mu\not{\mathcal{D}}\mathcal{D}^\mu\Big\}\frac{\Lambda^\delta-\mu^\delta}{\delta\,M_{(D)}^{\delta+2}}\Bigg]\psi\,.
\end{aligned}
\tag{5.12}
$$

From the counterterms (5.12) we can compute the gravitational contributions to the wavefunction and mass renormalisation of the fermion:

$$
\delta_\psi = \frac{1}{(4\pi)^{1+\delta/2}\Gamma\left(\frac{\delta}{2}+2\right)}\Bigg\{\frac{3(11+9\delta)}{(\delta+2)^2}\frac{\Lambda^{\delta+2}-\mu^{\delta+2}}{M_{(D)}^{\delta+2}}+\frac{19\delta-32}{8\delta}m_\psi^2\frac{\Lambda^\delta-\mu^\delta}{M_{(D)}^{\delta+2}}\Bigg\}
\tag{5.13}
$$

$$\delta_{m_\psi} = \frac{1}{(4\pi)^{1+\delta/2}\Gamma\left(\frac{\delta}{2}+2\right)}\left\{\frac{25\delta+46}{2(\delta+2)^2}\frac{\Lambda^{\delta+2}-\mu^{\delta+2}}{M_{(D)}^{\delta+2}} + \frac{15}{8}m_\psi^2\frac{\Lambda^\delta-\mu^\delta}{M_{(D)}^{\delta+2}}\right\} \tag{5.14}$$

The mass renormalisation was defined in (4.25) as $m_\psi\delta_{m_\psi} = \delta_{\overline{\psi}\psi} - m_\psi\delta_\psi$.

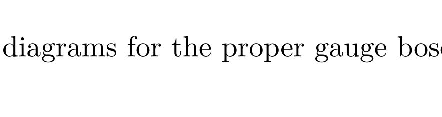
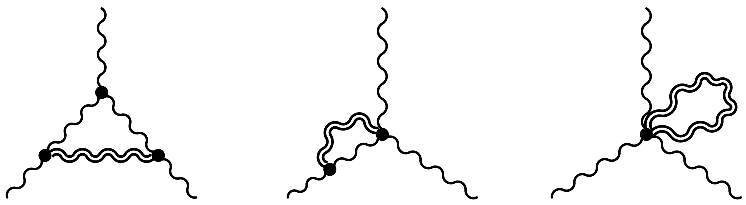


Figure 5.6.: One-loop diagrams for the proper gauge boson two-point function.



+ 3 permutations   + 3 permutations

Figure 5.7.: One-loop diagrams for the proper gauge boson three-point vertex.

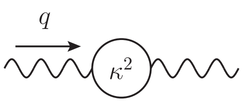
5.4. Counterterms involing only the Gauge Field

As the reader might already know form [15, 17], the results for the gravitational renormalisation of the gluon field in $4+\delta$ dimensions are zero for most quantities. The only term renormalised by gravitational effects in $4+\delta$ dimensions is the higher derivative term of the Lee-Wick form

$$\mathcal{L}_{\mathrm{YM}}^{\mathrm{c.t.}\,(4+\delta)} = \frac{1}{(4\pi)^{1+\delta/2}\Gamma\left(\frac{\delta}{2}+1\right)} \frac{8}{3\delta} \frac{\Lambda^{\delta}-\mu^{\delta}}{M_{(D)}^{\delta+2}} \operatorname{tr}\left\{\mathcal{D}^{\mu}F_{\mu\rho}\mathcal{D}_{\nu}F^{\nu\rho}\right\} . \tag{5.15}$$

Since the wavefunction renormalisation of the gluon is one of the major results of this work, we will not banish its full form for $d+\delta$ dimensions into the appendix, but present it more prominently a this point.

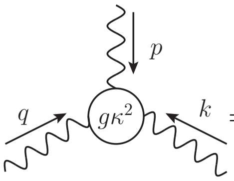
The diagrams in Figure 5.6 lead to the gravitational corrections to the gluonic two-point function which are



$$= \frac{i(q^2\eta^{\mu\nu}-q^{\mu}q^{\nu})}{(4\pi)^{D/2-1}\Gamma\left(\frac{D}{2}\right)M_{(D)}^{D-2}} \left[\frac{2(d-4)}{D-2}\Big((d-3)(d-2) + \frac{\delta}{d}(d^2-4d+8)\Big)\frac{\Lambda^{D-2}-\mu^{D-2}}{D-2} - q^2\frac{4(d-2)(d((2D+\delta)-6)-8\delta)}{d(d+2)}\frac{\Lambda^{D-4}-\mu^{D-4}}{D-4}\right] + \cdots \tag{5.16}$$

If we were only interested in Abelian gauge interactions, the point-two function would yield all needed information and we could end our calculation at this point. Photons are neutral and have no self coupling. Also, gravity cannot renormalise a photonic vertex term at order $\kappa^2 \sim G_{\mathrm{Newton}}$. To renormalise the simplest interaction operator $(F_{\mu\nu}F^{\mu\nu})^2$ one needs to go to order G_{Newton}^2. These operators are not of interest for our aims as they are suppressed in terms of the energy expansion.

The second higher derivative term $\operatorname{tr}[F^3]$ only exists for non Abelian gauge fields and cannot be constructed out of Maxwell fields. Since it is cubic in the fieldstrength tensor, it contributes at tree level only to amplitude of at least three gauge bosons:



$$= g f^{abc}\Big[\eta^{\mu\nu}\big((p^\rho(2\Delta_1 p\cdot q + \Delta_1 p\cdot k \tag{5.17}$$

$$+ (3\Delta_1 - \tfrac{3}{2}\Delta_2) q\cdot k) - q^\rho(\dots)) + \cdots$$

$$- \Delta_1\big(k^\mu k^\nu (p^\rho - q^\rho) + \cdots\big) \tag{5.18}$$

$$- (3\Delta_1 - \tfrac{3}{2}\Delta_2)(p^\rho q^\mu k^\nu - p^\nu q^\rho k^\mu)\Big].$$

with

$$\Delta_1 = \frac{-1}{(4\pi)^{D/2-1}\Gamma\left(\frac{D}{2}\right) M_{(D)}^{D-2}} \frac{8(d-2)}{(d+2)} \left(3 - d + \frac{\delta}{d}\left(4 - \tfrac{3}{2}d\right)\right) \frac{\Lambda^{D-4} - \mu^{D-4}}{D-4}$$

$$\Delta_2 = \frac{-1}{(4\pi)^{D/2-1}\Gamma\left(\frac{D}{2}\right) M_{(D)}^{D-2}} \frac{16(d-4)\left(8 - 8d - d^2 - \delta(16+d)\right)}{d(d+2)(d-2)} \frac{\Lambda^{D-4} - \mu^{D-4}}{D-4}.$$

These divergences can be absorbed by the counterterms proportional to the Yang-Mills action and the four derivative operators (3.16). The Feynman rules of these operators are given in (3.18) and (3.19). The counterterms which cancel the gravitational one-loop divergences are

$$\begin{aligned}\mathcal{L}_{\mathrm{YM}}^{\mathrm{c.t.}} = \frac{1}{(4\pi)^{D/2-1}\Gamma\left(\frac{D}{2}\right) M_{(D)}^{D-2}} \Bigg\{ &-\frac{1}{2}\frac{2(d-4)}{D-2}\Big((d-3)(d-2) \\ &+ \frac{\delta}{d}(d^2 - 4d + 8)\Big) \operatorname{tr}\left[\hat F_{\mu\nu}\hat F^{\mu\nu}\right] \frac{\Lambda^{D-2} - \mu^{D-2}}{D-2} \\ &+ \Bigg(\frac{8(d-2)}{(d+2)}\left(3 - d + \frac{\delta}{d}\left(4 - \tfrac{3}{2}d\right)\right) \operatorname{tr}\left[D^\mu \hat F_{\mu\rho} D_\nu \hat F^{\nu\rho}\right] \\ &+ \frac{16(d-4)\left(8 - 8d - d^2 - \delta(16+d)\right)}{d(d+2)(d-2)} g \operatorname{tr}\left[\hat F^\alpha_\beta \hat F^\beta_\gamma \hat F^\gamma_\alpha\right]\Bigg) \frac{\Lambda^{D-4} - \mu^{D-4}}{D-4}\Bigg\},\end{aligned} \tag{5.19}$$

where $\hat F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i g^{\mathrm{ren.}}[A_\mu, A_\nu]$ is the fieldstrength tensor defined with the one-loop renormalised coupling $g^{\mathrm{ren.}}$. From the counterterms (5.15) we can read off the gravitational contributions to the wavefunction renormalisation of the gauge field:

$$\delta_A = \frac{2}{(4\pi)^{D/2-1}\Gamma\left(\frac{D}{2}\right) M_{(D)}^{D-2}} \frac{d-4}{D-2}\left((d-3)(d-2) + \frac{\delta}{d}(d^2 - 4d + 8)\right) \frac{\Lambda^{D-2} - \mu^{D-2}}{D-2}. \tag{5.20}$$

6. Renormalisation of the Couplings

In the last chapter, we calculated the wavefunction and mass renormalisation of the fields as well as the renormalisation of their higher derivative operators. Now we can use our results in the computation of the running of the Standard Model couplings. For the gravitational contributions to the renormalisation of the gauge coupling constants g we need no further input. We can exploit gauge invariance to determine its running from the wavefunction renormalisation of the gluon. In order to compute the gravitational renormalisation of the quartic scalar coupling λ and the Yukawa coupling $\mathcal{y}$, we need to compute the corresponding one-loop interaction diagrams.

In the first section, we calculate the gravity induced renormalisation of the Yang-Mills and Maxwell coupling constants. The calculation was presented before in [15] for the four dimensional case and in [17] for the general $d+\delta$ dimensional case. The results for the quartic scalar self-interaction and the Yukawa coupling were only published for the four dimensional case in dimensional regularisation, [32, 58]. In sections 6.2 and 6.3 we give the full result in $d+\delta$ dimensions using the cut-off regularisation established in chapter 4.

6.1. Renormalisation of gauge couplings

To calculate the gravitational contribution to the running of the Maxwell and Yang-Mills coupling constants only the gauge boson two point function is needed. Due to the universality of the gauge coupling g, its renormalisation constants obtained from the three-gluon and gluon-ghost vertex must be the same, i. e.

$$\frac{Z_{A^3}}{{Z_A}^{3/2}} = \frac{Z_{\bar{c}Ac}}{{Z_A}^{1/2} Z_{\mathrm{gh}}}\,, \tag{6.1}$$

where $Z_{A^3}(Z_{\bar{c}Ac})$ denote the vertex renormalisation constants for gluon (ghost) couplings. Since the gluon ghosts are introduced after the expansion of the metric and thus do not couple to gravitons, we have with $Z_{\bar{c}Ac} = 1 + \delta_{\bar{c}Ac}$ and $Z_{\mathrm{gh}} = 1 + \delta_{\mathrm{gh}}$ that

$$\delta_{\bar{c}Ac}\big|_{\mathcal{O}(\kappa^2)} = \delta_{\mathrm{gh}}\big|_{\mathcal{O}(\kappa^2)} = 0 \tag{6.2}$$

and so by virtue of (6.1) at one-loop level

$$\delta_{A^3}\Big|_{\mathcal{O}(\kappa^2)} = \delta_A\Big|_{\mathcal{O}(\kappa^2)}\,. \tag{6.3}$$

This identity was also discussed for the fermionc counterterms in section 4.3 and used to derive the appropriate parametrisation of bubble integrals.

We can simplify the formula for the gravitational part of the Yang-Mills β function:

$$\beta_g = g\frac{\partial}{\partial \log \mu}\left(\tfrac{3}{2}\delta_A - \delta_{A^3}\right) \tag{6.4}$$

$$\Rightarrow \quad \beta_g\Big|_{\mathcal{O}(\kappa^2)} = \frac{g}{2}\frac{\partial}{\partial \log \mu}\delta_A \tag{6.5}$$

From (5.15) we have

$$\delta_A = \frac{2}{(4\pi)^{D/2-1}\Gamma\left(\frac{D}{2}\right)M_{(D)}^{D-2}}\frac{d-4}{D-2}\Big((d-3)(d-2) + \frac{\delta}{d}(d^2-4d+8)\Big)\frac{\Lambda^{D-2}-\mu^{D-2}}{D-2} \tag{6.6}$$

and the β function is thus

$$\beta_g = -\frac{g}{(4\pi)^{D/2-1}\Gamma(\frac{D}{2})}\frac{d-4}{D-2}\left((d-3)(d-2)+\frac{\delta}{d}(d^2-4d+8)\right)\frac{\mu^{D-2}}{M_{(D)}^{D-2}}\,. \tag{6.7}$$

Irrespective of the presence of higher dimensional gravity, for $d=4$ (3-branes) there are no gravitational corrections to the Yang-Mills β function at one-loop order. It is crucial that the vanishing of the leading gravitational divergence and thus the absence of a gravitationally induced running of the coupling constant is a unique feature of four dimensional gauge theories, independent of the number of extra dimensions.

6.2. Renormalisation of the Quartic Scalar Interaction

For the non-gauge interactions, i. e., scalar self-interaction and Yukawa coupling, we cannot exploit symmetry principles to simplify the calculation. We have to explicitly compute the vertex corrections. We can however use the disentanglement of the one-loop flow and consider only the coupling of a neutral, real scalar field and a Dirac fermion.

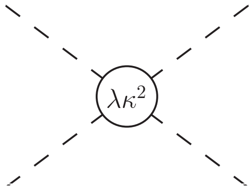
Thus, to determine the gravitational renormalisation of the quartic scalar self-coupling λ we only have to investigate one real scalar coupled to gravity:

$$\mathcal{L} = \tfrac{2}{\kappa^2}\sqrt{-g}R + \sqrt{-g}\left[\tfrac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \tfrac{1}{2}m_\varphi^2\varphi^2 - \frac{\lambda}{4!}\varphi^4\right]\,. \tag{6.8}$$

The gravitational wavefunction renormalisation for the real scalar field is the same as for the complex one. So, we can use the results from chapter 5 and Appendix A, equations (5.8) and (A.5):

$$\begin{aligned}\delta_\varphi = {} & \frac{1}{(4\pi)^{D/2-1}\Gamma\left(\frac{D}{2}\right)M_{(D)}^{D-2}}\Bigg(\frac{2(d-2)(d^2(D-3)+\delta(2-d))}{d(D-2)}\frac{\Lambda^{D-2}-\mu^{D-2}}{D-2} \\ & -\frac{4d((2d(3-\delta)+11\delta)-20)-40\delta+64}{d(d-2)}m_\varphi^2\frac{\Lambda^{D-4}-\mu^{D-4}}{D-4}\Bigg) \\ \xrightarrow{d=4} {} & \frac{1}{(4\pi)^{1+\delta/2}\Gamma\left(\frac{\delta}{2}+2\right)}\left\{\frac{2(8+5\delta)}{(\delta+2)^2}\frac{\Lambda^{\delta+2}-\mu^{\delta+2}}{M_{(D)}^{\delta+2}} - \frac{16+\delta}{\delta}m_\varphi^2\frac{\Lambda^{\delta}-\mu^{\delta}}{M_{(D)}^{\delta+2}}\right\}.\end{aligned} \tag{6.9}$$

In addition we need the gravitational one-loop correction to the quartic scalar amplitude Figure 6.1:



$$= \frac{i\,\lambda}{(4\pi)^{D/2-1}\Gamma\left(\frac{D}{2}\right) M_{(D)}^{D-2}}\left[\frac{2d(d(D-1)-2)}{D-2}\frac{\Lambda^{D-2}-\mu^{D-2}}{D-2} + \frac{16d(\delta-2)}{d-2}m_\varphi^2\frac{\Lambda^{D-4}-\mu^{D-4}}{D-4}\right] + \cdots \quad (6.10)$$

Note that we discard all terms which depend on the external momenta. These correspond to operators with additional space-time derivatives, e. g., $\varphi^2\partial_\mu\varphi\partial^\mu\varphi$, and do not contribute to the renormalisation of λ.

With these ingredients, we can use equation (4.32)

$$\beta_\lambda = \lambda\frac{\partial}{\partial\log\mu}\left(2\delta_\varphi - \delta_{\varphi^4}\right)$$

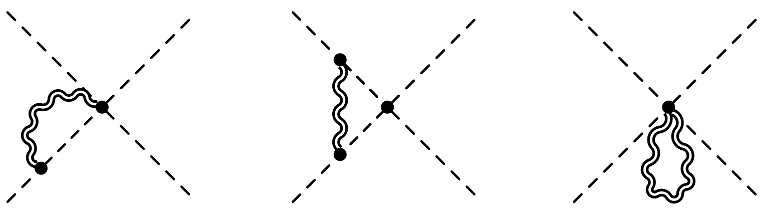
to calculate the β function for the scalar self-interaction

$$\beta_\lambda = \frac{\lambda}{(4\pi)^{D/2-1}\Gamma\left(\frac{D}{2}\right) M_{(D)}^{D-2}}\left(\frac{16\delta - 2d(d-2)(d(D-7)-6\delta)}{d(D-2)}\mu^{D-2} + \frac{8\left(((11-2d)d-10)\delta^2 + (17d-18)(d-2)\delta + 2(d(d-10)+8)(d-2)\right)}{d(d-2)(D-2)}m_\varphi^2\mu^{D-4}\right). \quad (6.11)$$

This result becomes n $4+\delta$ dimensions

$$\beta_\lambda = \frac{\lambda}{(4\pi)^{1+\delta/2}\Gamma\left(\frac{\delta}{2}+2\right) M_{(D)}^{\delta+2}}\left(\frac{12(\delta+4)}{\delta+2}\mu^{2+\delta} + \frac{2\left(\delta^2+50\delta-32\right)}{\delta+2}m_\varphi^2\mu^\delta\right) \quad (6.12)$$

The dominant term $\propto \mu^{2+\delta}$ is positive for all values of δ including the standard scenario without extra dimensions $\delta = 0$. This is in contrast to the Yang-Mills β function whose gravitational one-loop part is in all scenarios at least non-positive [25], thereby (if non-zero)



+ 3 permutations + 5 permutations

Figure 6.1.: Order $\lambda\kappa^2$ corrections to the φ^4 interaction.

acting in direction of asymptotic freedom. Hence, the leading term in perturbative gravity does not qualitatively change the behaviour of the running of the coupling.

In [32] we used dimensional regularisation in four dimensions which only captured the logarithmic divergences. The part of the gravitational contributions to the β function arising from the logarithmic divergences is reproduced by our calculation with cut-off regularisation[1].

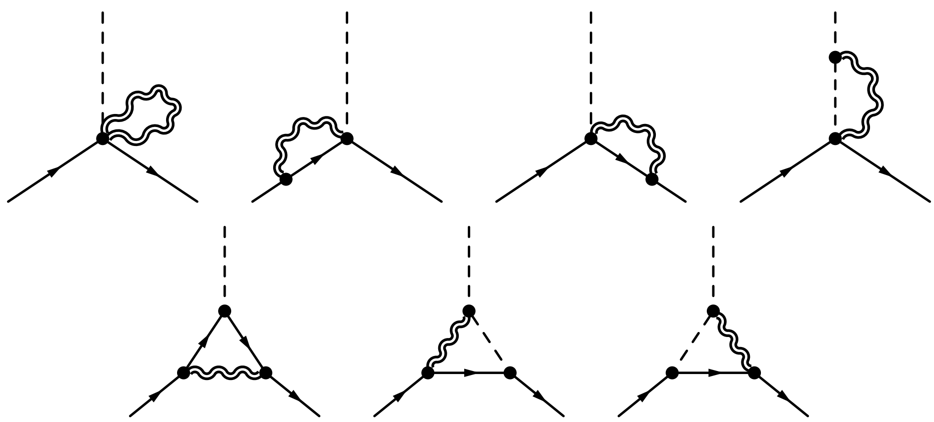


Figure 6.2.: Order $y\kappa^2$ corrections to the Yukawa interaction.

6.3. Renormalisation of the Yukawa Coupling

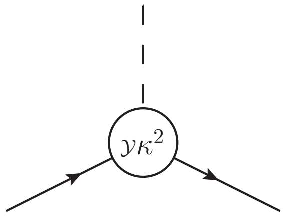
The toy model we use to compute gravity's effect on the one-loop running of the Yukawa interaction consists of one real scalar and one Dirac fermion:

$$\mathcal{L} = \tfrac{2}{\kappa^2}\sqrt{-g}R + \sqrt{-g}\left[\overline{\psi}(i\not{\mathcal{D}} - m_\psi)\psi + \tfrac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \tfrac{1}{2}m_\varphi^2\varphi^2 - y\,\overline{\psi}\varphi\psi\right]. \qquad (6.13)$$

We already cited the wavefunction renormalisation of the scalar in the last section, (6.9). In addtion, we need the gravitational part of fermion wavefunction corrections to compute the scale dependence of the Yukawa coupling. These were also determined in chapter 5, equation (5.13). Again, we use the full result in $d+\delta$ dimensions, (A.10):

$$\begin{aligned}
\delta_\psi = {} & \frac{1}{(4\pi)^{D/2-1}\Gamma\left(\frac{D}{2}\right)M_{(D)}^{D-2}}\Bigg(\frac{d-1}{2d(D-2)}(8-4\delta+d(d(4D-7) \\
& \qquad + 3\delta - 16))\frac{\Lambda^{D-2}-\mu^{D-2}}{D-2} \\
& + \frac{(10\delta + d(18-5\delta+d(d+3\delta-11))) - 24}{2d(d-2)}m_\psi^2\frac{\Lambda^{D-4}-\mu^{D-4}}{D-4}\Bigg) \\
\xrightarrow{d=4} {} & \frac{1}{(4\pi)^{1+\delta/2}\Gamma\left(\frac{\delta}{2}+2\right)}\left\{\frac{3(11+9\delta)}{(\delta+2)^2}\frac{\Lambda^{\delta+2}-\mu^{\delta+2}}{M_{(D)}^{\delta+2}} + \frac{19\delta-32}{8\delta}m_\psi^2\frac{\Lambda^\delta-\mu^\delta}{M_{(D)}^{\delta+2}}\right\}.
\end{aligned}$$

All that is left to calculate are the gravitational vertex corrections in Figure 6.2 to the fermion–scalar interaction:



$$\begin{aligned}
= {} & \frac{i\,y}{(4\pi)^{D/2-1}\Gamma\left(\frac{D}{2}\right)M_{(D)}^{D-2}}\Bigg[\Bigg\{\frac{4(\delta-2)}{d-2}m_\varphi^2 \\
& + \frac{3((5\delta+d(-5+d+3\delta))-12)}{2(d-2)}m_\psi^2\Bigg\}\frac{\Lambda^{D-4}-\mu^{D-4}}{D-4} \\
& + \frac{d((d(-5+4D)+5\delta)-19)-5\delta+12}{2(D-2)}\frac{\Lambda^{D-2}-\mu^{D-2}}{D-2}\Bigg] + \cdots \qquad (6.14)
\end{aligned}$$

Again, we discard all terms which depend on the external momenta as in the calculation of the running of the scalar coupling. The reason is the same as in the calculation of the gravitational correction to the scalar self-interaction.

Now, we can use equation (4.33)

$$\beta_y = y\frac{\partial}{\partial\log\mu}\left(\delta_\psi + \tfrac{1}{2}\delta_\varphi - \delta_{\overline{\psi}\varphi\psi}\right)$$

[1]Note that $\kappa^2 = 1/(32\pi M_{\text{Planck}}^2)$ was used in [32] as the gravitational coupling constant.

to calculate the gravitational part of the one-loop Yukawa β function:

$$\beta_{\mathcal{Y}} = \frac{\mathcal{Y}}{(4\pi)^{D/2-1}\Gamma\left(\frac{D}{2}\right)M_{(D)}^{D-2}}\left(\frac{2(\delta+2)-d(d(d(D-8)-7\delta+11)+5\delta+6)}{d(D-2)}\mu^{D-2}\right.$$
$$-\frac{2\left((d(2d-11)+10)\delta^2+(d(2d-19)+18)(d-2)\delta-2(3d(d-4)+8)(d-2)\right)}{d(d-2)(D-2)}m_\varphi^2\mu^{D-4}$$
$$+\frac{1}{2d(d-2)(D-2)}\Big(2(-6+d)d(-11+d(2+d))+44\delta+d(-68+d(19+5d))\delta$$
$$\left.+(-10+(5-3d)d)\delta^2-48\Big)m_\psi^2\mu^{D-4}\right). \quad (6.15)$$

In $4+\delta$ dimension it becomes

$$\beta_{\mathcal{Y}} = \frac{\mathcal{Y}}{(4\pi)^{1+\delta/2}\Gamma\left(\frac{\delta}{2}+2\right)M_{(D)}^{\delta+2}}\left(\frac{15}{2}\mu^{2+\delta}\right.$$
$$\left.+\left\{\frac{\delta^2+26\delta+16}{2(\delta+2)}m_\varphi^2-\frac{19\delta^2-198\delta+128}{8(\delta+2)}m_\psi^2\right\}\mu^\delta\right). \quad (6.16)$$

Again, the leading term in the energy expansion $\propto \mu^{2+\delta}$ is positive, independently of the number of extra dimensions δ.

7. Functional Renormalisation

Until now, we applied the well established and widely used methods of perturbative field theory and already the one-loop calculations have proven to be quite involved. The question is how we could continue our calculations. The natural step would be to go to two loops. Such a calculation has the main obstacle that our regularisation method will not work at higher loops, as has been discussed already at the end of section 4.3. Consequently, we have to completely revise our methods for a deeper analysis. Or better, we should use a different ansatz.

Instead of continuing the perturbation expansion, we decided approach the question using a functional renormalisation method, namely by calculating the the flow equation of the effective average action. Unfortunately, we were not able to complete our calculations within this thesis.

We will give a short introduction to the formalism of functional renormalisation in section 7.1. The next section deals with the treatment of gauge theories in the background field formalism with a special focus on the Einstein-Yang-Mills system. In these introductive sections we partly follow the lines of [59, 60]. Finally in section 7.3, we present some of the FORM routines we developed for background field calculations.

The set-up in this chapter differs from the previous ones. Instead of a d dimensional matter brane in a D dimensional universe, we simply consider a d dimensional world. To avoid ambiguities which can arise from the Wick rotation, we do our calculations in a Euclidean manifold from the start. Recent analyses [61] shows that the results in pure gravity does not differ qualitatively from the Lorentzian case. We assume this holds true also for the Einstein-Yang-Mills system.

Throughout this chapter, we will use DeWitt's condensed index notation. Lower case Latin indices $i, j, \dots$ label the field type (e. g. metric $g_{\mu\nu}$ or gauge boson A^a_μ) as well as the space-time coordinate. Thus, contraction of indices implies integration over the space-time, i. e.,

$$J_i \varphi^i = \int \mathrm{d}^d x J_I(x) \varphi^I(x) \,,$$

where the capital letters I only specify the field type.

7.1. The effective average action

To start, we will give a short derivation of the exact functional renormalisation group equation. We will not define the path integral which is usually done by discretisation in lattice field theory or Gaussian integrals in perturbative quantum field theory. We derive a functional differential equation for the so called effective average action from the assumption that a path integral representation of the field theory exists. This functional renormalisation group or Wetterich equation together with boundary conditions should be understood as the definition of the effective average action and thereby of the quantum theory.

The generating functional of the connected Green's functions is defined as

$$\exp W[J] = \int \mathcal{D}\varphi \exp\{-S[\varphi] + \varphi^i J_i\}\,, \tag{7.1}$$

where J_i are external sources coupling to the fields and the path integral $\int \mathcal{D}\varphi$ is taken over the whole space of field configurations. By Legendre transformation, we obtain the effective action

$$\Gamma[\phi] = \phi^i J_i[\phi] - W[J[\phi]] \tag{7.2}$$

which depends on the field expectation values ϕ^i. The currents $J_i[\phi]$ are obtained via inverting $\phi^i = \frac{\delta W}{\delta J_i}$ or equivalently from $J_i = \frac{\Gamma}{\phi^i}$. The effective action governs the dynamics of the expectation values of the fields and encodes all informations from quantum effect.

Now, we want to use the renormalisation group concepts to circumvent the direct evaluation of the path integral, i. e., integrating out modes of the quantum fluctuations momentum shell by momentum shell. In order to do so, we modify (7.1) by adding a infra red cut-off action to the action S

$$\exp W_\mathrm{k}[J] = \int \mathcal{D}\varphi \exp\{-S[\varphi] - \Delta S_\mathrm{k}[\varphi] + \varphi^i J_i\}\,. \tag{7.3}$$

The cut-off operator is quadratic in the fields and has the form

$$\Delta S_\mathrm{k}[\varphi] = \tfrac{1}{2} \int \frac{\mathrm{d}^d p}{(2\pi)^d} \varphi(-p) \mathcal{R}_\mathrm{k}(p^2) \varphi(p)\,. \tag{7.4}$$

We demand on the cut-off kernel $\mathcal{R}_\mathrm{k}(p^2)$ to have the asymptotic behaviour

$$\mathcal{R}_\mathrm{k}(p^2) \approx \begin{cases} \mathrm{k}^2 & \text{for} \quad p^2 \ll \mathrm{k}^2 \\ 0 & \text{for} \quad p^2 \gg \mathrm{k}^2\,. \end{cases} \tag{7.5}$$

Hence, the functional integral is suppress for all modes with momenta smaller then k. The quantum modes with momenta higher then k contribute without any suppression. We can write the cut-off action using a cut-off operator $\mathcal{R}_{\mathrm{k}\,ij}$ in DeWitt's condensed notation

$$\Delta S_\mathrm{k}[\varphi] = \tfrac{1}{2} \varphi^i \mathcal{R}_{\mathrm{k}\,ij}(-\partial^2) \varphi^j\,. \tag{7.6}$$

We write the momentum dependence of the cut-off operator by replacing p^2 in the argument of the cut-off kernel by the Laplacean $-\partial^2$.

We again preform a Legendre transform

$$\tilde{\Gamma}_\mathrm{k}[\phi] = \phi^i J_{\mathrm{k}\,i} - W_\mathrm{k}[J]\,. \tag{7.7}$$

Now, the currents $J_{\mathrm{k}\,i}$ depend on the cut-off scale k since they are derived from W_k via $\phi^i = \frac{\delta W_\mathrm{k}}{\delta J_i}$. The k dependence of the currents chosen such that it cancels the scale dependence of the generating functional W_k yielding scale independent fields ϕ^i.

The actual effective average action $\Gamma_\mathrm{k}[\phi]$ is obtain by subtracting $\Delta S_\mathrm{k}[\phi]$ from the Legendre transformed functional $\tilde{\Gamma}_\mathrm{k}[\phi]$:

$$\Gamma_\mathrm{k}[\phi] = \tilde{\Gamma}_\mathrm{k}[\phi] - \Delta S_\mathrm{k}[\phi] = \tilde{\Gamma}_\mathrm{k}[\phi] - \tfrac{1}{2} \phi^i \mathcal{R}_{\mathrm{k}\,ij} \phi^j. \tag{7.8}$$

The motivation for this subtraction is that the renormalisation group equation for $\Gamma_\mathrm{k}[\phi]$ is simpler then the equation for $\tilde{\Gamma}_\mathrm{k}[\phi]$.

To determine the scale dependence of the effective average action $\Gamma_\mathrm{k}[\phi]$, we first take the derivative with respect to k of (7.7) with the definition (7.3) inserted:

$$\begin{aligned} \mathrm{k}\partial_\mathrm{k}\tilde{\Gamma}_\mathrm{k}[\phi] &= \phi^i \mathrm{k}\partial_\mathrm{k} J_{\mathrm{k}\,i} - e^{-W_\mathrm{k}} \int \mathcal{D}\varphi e^{(-S-\Delta S_\mathrm{k}+\varphi^i J_{\mathrm{k}\,i})} \left(-\frac{1}{2}\varphi^i \mathrm{k}\partial_\mathrm{k}\mathcal{R}_{\mathrm{k}\,ij}\varphi^j + \varphi^i \mathrm{k}\partial_\mathrm{k} J_{\mathrm{k}\,i} \right) \\ &= \frac{1}{2}\left\langle \varphi^i\varphi^j \right\rangle \mathrm{k}\partial_\mathrm{k}\mathcal{R}_{\mathrm{k}\,ij} \,. \end{aligned}$$

We can further use, that

$$\left(\frac{\delta^2\tilde{\Gamma}_\mathrm{k}}{\delta\phi^i\delta\phi^j} \right)^{-1} = \left\langle \varphi^i\varphi^j \right\rangle - \phi^i\phi^j$$

and the definition (7.8) to arrive at the exact functional renormalisation group or Wetterich equation [62]

$$\partial_\mathrm{t}\Gamma_\mathrm{k}[\phi] = \frac{1}{2}\left(\Gamma^{(2)}_\mathrm{k} + \mathcal{R}_\mathrm{k} \right)^{-1}_{ij} \partial_\mathrm{t}\mathcal{R}_{\mathrm{k}\,ij} \tag{7.9}$$

for the effective average action. Here, we use the short-hand notation for the Hassian:

$$\Gamma^{(2)}_{\mathrm{k}\,ij} = \frac{\delta^2\Gamma_\mathrm{k}}{\delta\phi^i\delta\phi^j} \tag{7.10}$$

and the dimensionless parameter $\mathrm{t} = \ln(\mathrm{k}/\mathrm{k}_0)$ which is called renormalisation group time. It is easy to see, that the effective average action interpolates between the bare and the effective action, i. e.,

$$\lim_{\mathrm{k}\to\infty} \Gamma_\mathrm{k}[\phi] = S[\phi] \tag{7.11}$$

$$\lim_{\mathrm{k}\to 0} \Gamma_\mathrm{k}[\phi] = \Gamma[\phi]\,. \tag{7.12}$$

For a renormalisable theory (7.9) together with the boundary condition (7.11) defines the effective average action.

7.2. Gauge theories and background field formalism

Let us consider a local, non-trivial symmetry transformation $\mathcal{G}^{\omega^\alpha}$ of (some of) the fields

$$\varphi^i \to \mathcal{G}^{\omega^\alpha}\varphi^i \neq \varphi^i\,. \tag{7.13}$$

We parametrise the symmetry transformation by a set of local functions $\{\omega^\alpha\}$. If the action $S[\varphi]$ is invariant under the transformation, i. e.,

$$S[\varphi] \to S[\mathcal{G}^{\omega^\alpha}\varphi] = S[\varphi] \tag{7.14}$$

we call the transformation $\mathcal{G}$ gauge transformation and the field theory defined by S gauge theory. Typical gauge theories are the Yang-Mills theory and gravity.

A set of field configurations which are connected by gauge transformations is called gauge orbit. Each gauge orbit can be represented by one of its elements $\varphi^i_{\text{ref.}}$:

$$\left[\varphi^i_{\text{ref.}}\right]^{\text{orbit}} = \{\varphi^i | \varphi^i = \mathcal{G}^{\omega^\alpha}\varphi^i_{\text{ref.}}\}\,. \tag{7.15}$$

The change of the field variables under infinitesimal gauge transformation with the parameter ϵ^α can be written as

$$\mathcal{G}^{\epsilon^\alpha}\varphi^i = \varphi^i + \theta^i_\alpha[\varphi]\epsilon^\alpha, \tag{7.16}$$

where the $\theta^i_\alpha[\varphi]$ are local operators.

If we naively used the standard path integral

$$\int \mathcal{D}\varphi \exp\{-S[\varphi]\} \tag{7.17}$$

to quantise gauge theories, we would fail. The measure $\mathcal{D}\varphi$ contains all field configurations, especially multiple configurations of each gauge orbit which are physically equivalent. Hence, the path integral would lead to ill-defined quantities.

In order to define a path integral for a gauge theory, we have to pick one field configuration from each gauge orbit. This is achieved by integrating only over field configurations which satisfy a linear gauge fixing condition:

$$F^\alpha[\varphi] = \mathcal{F}^\alpha_i\varphi^i = 0\,. \tag{7.18}$$

We can now use the Faddeev-Popov trick and replace the naive generating functional (7.1) by the gauge-fixed functional

$$\exp W[J] = \int \mathcal{D}\varphi \det\left(\mathcal{F}^\alpha_i\theta^i_\beta\right)\delta\left(\mathcal{F}^\alpha_i\varphi^i\right)\exp\{-S[\varphi] + \varphi^i J_i\}\,. \tag{7.19}$$

The Faddeev-Popov determinant $\det\left(\mathcal{F}^\alpha_i\theta^i_\beta\right)$ and gauge fixing $\delta\left(\mathcal{F}^\alpha_i\varphi^i\right)$ can be brought into the exponent which yields the famous Faddeev-Popov action:

$$\exp W[J] = \int \mathcal{D}\varphi\mathcal{D}c\mathcal{D}\bar{c}\exp\{-\tilde{S}[\varphi,\bar{c},c] + \varphi^i J_i\} \tag{7.20}$$

$$\text{with } \tilde{S}[\varphi] = S[\varphi] + \frac{1}{2\xi}\int \mathcal{F}^\alpha_i\varphi^i\mathcal{F}^\alpha_j\varphi^j + \int \bar{c}^\alpha\mathcal{F}^\alpha_i\theta^i_\beta c^\beta \tag{7.21}$$

For the exponentiation of the determinant $\det\left(\mathcal{F}^\alpha_i\theta^i_\beta\right)$ we introduced the anti-commuting ghost fields c^α and anti-ghost fields $\bar{c}^\alpha$. The (anti-)ghost fields are not invariant under gauge transformations. They transform, e. g., in Yang-Mills gauge theories as fields in the adjoint representation and in gravity theories as contra-variant vectors. We also introduced the gauge parameter ξ.

For gauge theories it is especially practical to use the background field formalism [63]. Instead of quantising the full fields, we split the fields into an arbitrary, but fixed background fields $\bar{\phi}^i$ and quantum fluctuations χ^i

$$\varphi^i = \bar{\phi}^i + \chi^i\,. \tag{7.22}$$

We can now impose a background field dependent linear gauge fixing condition

$$F^\alpha[\bar{\phi},\chi] = \mathcal{F}^\alpha_i[\bar{\phi}]\chi^i = 0 \tag{7.23}$$

on the quantum fields χ^i. We use the same trick as before to define the generating functional:

$$\exp W[\bar{\phi}, J] = \int \mathcal{D}\chi \mathcal{D}c\mathcal{D}\bar{c} \exp\{-\tilde{S}[\bar{\phi}, \chi, \bar{c}, c] + \chi^i J_i\} \tag{7.24}$$

$$\text{with } \tilde{S}[\bar{\phi}, \chi, \bar{c}, c] = S[\bar{\phi} + \chi] + \frac{1}{2\xi} \int \mathcal{F}^\alpha_i[\bar{\phi}]\chi^i \mathcal{F}^\alpha_j[\bar{\phi}]\chi^j + \int \bar{c}^\alpha \mathcal{F}^\alpha_i[\bar{\phi}]\theta^i_\beta[\bar{\phi} + \chi]c^\beta \tag{7.25}$$

which now also depends on the background fields $\bar{\phi}^i$. The gauge transformation of the (anti-)ghost fields are the same as before.

$\tilde{S}[\bar{\phi}, \chi, \bar{c}, c]$ is invariant under background gauge transformations $\mathcal{G}_{\mathrm{BG}}$

$$\mathcal{G}^{\omega^\alpha}_{\mathrm{BG}}\bar{\phi}^i = \mathcal{G}^{\omega^\alpha}\bar{\phi}^i \qquad\qquad \mathcal{G}^{\omega^\alpha}_{\mathrm{BG}}\chi^i = \mathcal{G}^{\omega^\alpha}\varphi^i - \mathcal{G}^{\omega^\alpha}\bar{\phi}^i \tag{7.26}$$

which can be written as

$$\begin{aligned} \mathcal{G}^{\epsilon^\alpha}_{\mathrm{BG}}\bar{\phi}^i = \bar{\phi}^i + \theta^i_\alpha[\bar{\phi}]\epsilon^\alpha \qquad\qquad & \mathcal{G}^{\epsilon^\alpha}_{\mathrm{BG}}\chi^i = \chi^i + \hat{\theta}^i_\alpha[\bar{\phi}, \chi]\epsilon^\alpha \\ \text{with} \quad & \hat{\theta}^i_\alpha[\bar{\phi}, \chi] \equiv \theta^i_\alpha[\bar{\phi} + \chi] - \theta^i_\alpha[\bar{\phi}] \end{aligned} \tag{7.27}$$

for infinitesimal transformations. We call the variation of the quantum fields under background gauge transformations $\hat{\theta}^i_\alpha[\bar{\phi}, \chi]$. We can use background gauge transformations to split the ghost action into a quadratic part, only depending on the (anti-)ghosts and the background fields, and an interaction partial

$$\begin{aligned} S_{\mathrm{gh}}[\bar{\phi}, \chi, \bar{c}, c] &= \int \bar{c}^\alpha \mathcal{F}^\alpha_i[\bar{\phi}]\theta^i_\beta[\bar{\phi} + \chi]c^\beta = S^0_{\mathrm{gh}}[\bar{\phi}, \bar{c}, c] + S^{\mathrm{int.}}_{\mathrm{gh}}[\bar{\phi}, \chi, \bar{c}, c] \\ \text{with} \quad S^0_{\mathrm{gh}}[\bar{\phi}, \bar{c}, c] &= \int \bar{c}^\alpha \mathcal{F}^\alpha_i[\bar{\phi}]\theta^i_\beta[\bar{\phi}]c^\beta, \\ S^{\mathrm{int.}}_{\mathrm{gh}}[\bar{\phi}, \chi, \bar{c}, c] &= \int \bar{c}^\alpha \mathcal{F}^\alpha_i[\bar{\phi}]\hat{\theta}^i_\beta[\bar{\phi}, \chi]c^\beta. \end{aligned} \tag{7.28}$$

The functional renormalisation group equation for gauge theories can be derived in the same manner as before. We start by adding a IR regulator term in the definition the gauge fixed generating functional (7.24):

$$\exp W_{\mathrm{k}}[\bar{\phi}, J] = \int \mathcal{D}\chi \mathcal{D}c\mathcal{D}\bar{c} \exp\{-\tilde{S}[\bar{\phi}, \chi, \bar{c}, c] - \Delta S_{\mathrm{k}}[, \bar{\phi}, \chi, \bar{c}, c] + \chi^i J_i + \bar{J}_\alpha c^\alpha + \bar{c}^\alpha J_\alpha\}. \tag{7.29}$$

Here, we introduced additional currents $\bar{J}_\alpha$ and J_α coupling to the ghost and anti-ghost fields, respectively. In order to maintain the invariance of the generation functional with respect of the background gauge transformations (7.26), we have to use the background covariant Laplacean $-\bar{\Delta}$ in the regulator action. Thus, the cut-off operator now depends on the background fields:

$$\Delta S_{\mathrm{k}}[\bar{\phi}, \chi, \bar{c}, c] = \tfrac{1}{2}\chi^i \mathcal{R}_{\mathrm{k}\,ij}[\bar{\phi}]\chi^j + \bar{c}^\alpha \mathcal{R}_{\mathrm{k}\,\alpha\beta}[\bar{\phi}]c^\beta \tag{7.30}$$

Since the (anti-)ghost fields are propagating degrees of freedom, we have to add an IR regulator term for them as well.

The effective average action is obtained by a Legendre transformation of W_{k} with respect of the currents and subtracting the regulator action:

$$\Gamma_{\mathrm{k}}[\bar{\phi},\bar{\chi},\bar{\zeta},\zeta] = J_{\mathrm{k}\,i}\bar{\chi}^i + \bar{J}_{\mathrm{k}\,\alpha}\zeta^\alpha + \bar{\zeta}^\alpha J_{\mathrm{k}\,\alpha} - W_{\mathrm{k}}[\phi,J] - \Delta S_{\mathrm{k}}[\bar{\phi},\bar{\chi},\bar{\zeta},\zeta]\,. \tag{7.31}$$

Here, we call the expectation values of the quantum fields $\bar{\chi}^i$ and of the (anti-)ghosts $\bar{\zeta}^\alpha$, ζ^α, respectively. These are related to the currents via

$$\bar{\chi}^i = \frac{\delta W_{\mathrm{k}}[\bar{\phi},J]}{\delta J_i}, \qquad \bar{\zeta}^\alpha = \frac{\delta W_{\mathrm{k}}[\bar{\phi},J]}{\delta J_\alpha}, \qquad \zeta^\alpha = \frac{\delta W_{\mathrm{k}}[\bar{\phi},J]}{\delta \bar{J}_\alpha}.$$

In order to keep the formulas more compact and readable let us introduce the super-field $\eta^A = \{\bar{\chi}^i, \bar{\zeta}^\alpha, \zeta^\alpha\}$ with its index A labelling both, the fluctuations $\bar{\chi}^i$ and the (anti-)ghost fields $\bar{\zeta}^\alpha$ and ζ^α.

The scaling behaviour of the effective average action for gauge theories is again described by the Wetterich equation

$$\partial_{\mathrm{t}}\Gamma_{\mathrm{k}}[\bar{\phi},\eta] = \frac{1}{2}\left(\Gamma^{(0,2)}_{\mathrm{k}}[\bar{\phi},\eta] + \mathcal{R}_{\mathrm{k}}\right)^{-1}_{AB}\partial_{\mathrm{t}}\mathcal{R}_{\mathrm{k}\,AB} \tag{7.32}$$

with

$$\Gamma^{(0,2)}_{\mathrm{k}}[\bar{\phi},\eta]_{AB} = \frac{\delta^2\Gamma_{\mathrm{k}}[\bar{\phi},\eta]}{\delta\eta^A\delta\eta^B}. \tag{7.33}$$

As before, the fully integrated effective average action becomes the background effective action $\Gamma_{\mathrm{k}\to 0}[\bar{\phi},\eta] = \Gamma[\bar{\phi},\eta]$. From the full background effective action $\Gamma[\bar{\phi},\eta]$ we obtain the standard (not background dependent) effective action by setting the field fluctuations to zero, i. e.,

$$\Gamma[\phi,\eta=0] = \Gamma[\phi]\,. \tag{7.34}$$

Note, that $\Gamma_{\mathrm{k}}[\bar{\phi},\eta=0]$ is not sufficient to solve the exact functional renormalisation group equation. Thus, the identification (7.34) is not meaningful for finite values of k.

7.2.1. Einstein-Yang-Mills

The Einstein-Yang-Mills system in d dimensions is described by the action

$$S_{\mathrm{EYM}}[g_{\mu,\nu},A^a_\mu] = \int \mathrm{d}^d x\sqrt{g}\left(\frac{2}{\kappa^2}(2\Lambda - R) - \frac{1}{4g^2_{\mathrm{YM}}}g^{\mu\nu}g^{\rho\sigma}F^a_{\mu\rho}F^a_{\nu\sigma}\right). \tag{7.35}$$

We write the indices of the Yang-Mills gauge group explicitly, i. e., we use the colour component fields A^a_μ with $A_\mu = A^a_\mu T^a$ and $[T^a,T^b] = if^{abc}T^c$. The components of the fieldstrength tensor are $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc}A^b_\mu A^c_\nu$. We split the metric and the gauge bosons

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \qquad A^a_\mu = \bar{A}^a_\mu + a^a_\mu \tag{7.36}$$

into the fixed background $\bar{g}_{\mu\nu}$, $\bar{A}^a_\mu$ and the quantum fields $h_{\mu\nu}$, a^a_μ.

The action (7.35) is invariant under two local transformations: The Yang-Mills gauge transformations $\mathcal{G}^{\mathrm{YM}}$, parametrised by ϵ^a and diffeomorphism transformations $\mathcal{G}^{\mathrm{diff}}$, parametrised by ϵ^μ:

$$\begin{aligned} \mathcal{G}^{\mathrm{YM}} A^a_\mu &= A^a_\mu + D^{ab}_\mu \epsilon^b, & \mathcal{G}^{\mathrm{YM}} g_{\mu\nu} &= g_{\mu\nu}, \\ \mathcal{G}^{\mathrm{diff}} A^a_\mu &= A^a_\mu - (\nabla_\nu A^a_\mu)\epsilon^\nu - A^a_\nu \nabla_\mu \epsilon^\nu, & \mathcal{G}^{\mathrm{diff}} g_{\mu\nu} &= g_{\mu\nu} - \nabla_\mu \epsilon_\nu - \nabla_\nu \epsilon_\mu \,. \end{aligned} \tag{7.37}$$

In the transformation rule for the gauge boson we use the gauge covariant derivative in adjoint representation $D^{ab}_\mu = \delta^{ab}\nabla_\mu - f^{abc}A^c_\mu$.

Instead of the plain diffeomorphism $\mathcal{G}^{\mathrm{diff}}$ we use $\widetilde{\mathcal{G}^{\mathrm{diff}}}$ which is an diffeomorphism transformations with a simultaneous gauge transformation $\epsilon^a = A^a_\mu \epsilon^\mu$ [21, 64]. The simultaneous gauge transformation has obviously no effect on the metric, hence $\widetilde{\mathcal{G}^{\mathrm{diff}}} g_{\mu\nu} = \mathcal{G}^{\mathrm{diff}} g_{\mu\nu}$. The transformation of the gauge field A^a_μ on the other hand is significantly simplified:

$$\begin{aligned} \widetilde{\mathcal{G}^{\mathrm{diff}}} A^a_\mu &= A^a_\mu - (\nabla_\mu A^a_\nu)\epsilon^\nu - A^a_\nu \nabla_\mu \epsilon^\nu + D^{ab}_\mu (A^b_\nu \epsilon^\nu) \\ &= A^a_\mu - (\nabla_\nu A^a_\mu)\epsilon^\nu + (\nabla_\mu A^a_\nu)\epsilon^\nu - f^{abc} A^c_\mu A^b_\nu \epsilon^\nu = A^a_\mu + F^a_{\mu\nu}\epsilon^\nu \end{aligned}$$

The infinitesimal transformation are generated by the transformation operators

$$\begin{aligned} \theta^{b\, A^a_\mu} &= D^{ab}_\mu, & \theta^{a\, g_{\mu\nu}} &= 0, \\ \theta^{A^a_\mu}_\rho &= F^a_{\mu\rho}, & \theta^{g_{\mu\nu}}_\rho &= -g_{\nu\rho}\nabla_\mu - g_{\mu\rho}\nabla_\nu. \end{aligned} \tag{7.38}$$

The change of the quantum fields under background gauge transformations is given by the following transformation operators

$$\begin{aligned} \hat\theta^{b\, a^a_\mu} &= -f^{abc} a^c_\mu & \hat\theta^{a\, h_{\mu\nu}} &= 0 \\ \hat\theta^{a^a_\mu}_\rho &= \bar D^{ab}_\mu \left(a^b_\rho\right) - \bar D^{ab}_\rho \left(a^b_\mu\right) + f^{abc} a^b_\mu a^c_\rho & \hat\theta^{h_{\mu\nu}}_\rho &= -h_{\nu\rho}\bar\nabla_\mu - h_{\mu\rho}\bar\nabla_\nu - \bar\nabla_\rho\left(h_{\mu\nu}\right), \end{aligned} \tag{7.39}$$

where we introduced the background covariant derivatives $\bar\nabla_\mu$ and $\bar D^{ab}_\mu$.

The gauge fixing conditions we impose to quantise the Einstein-Yang-Mills theory are

$$\begin{aligned} F^a &= \tfrac{1}{g_{\mathrm{YM}}} \bar g^{\mu\nu} \bar D^{ab}_\mu a^b_\nu = 0 \\ F_\rho &= \tfrac{\sqrt{2}}{\kappa} \bar g^{\mu\nu} \left(\bar\nabla_\mu h_{\nu\rho} - \frac{1+\alpha}{2} \bar\nabla_\rho h_{\mu\nu} + \beta\, \kappa^2 \bar F^a_{\rho\mu} a^a_\nu \right) = 0 \,. \end{aligned} \tag{7.40}$$

Here, we introduced two additional parameter α and β to cover a wider range of possible gauges. The choice $\alpha = 0 = \beta$ corresponds to the background formalism version of the harmonic gauge we used in the previous chapters. From the gauge fixing conditions we can read off the gauge fixing operators $\mathcal{F}^i_\alpha$ for the gluon and graviton fields:

$$\begin{aligned} \mathcal{F}^a_{a^b_\mu} &= \tfrac{1}{g_{\mathrm{YM}}} \bar g^{\mu\nu} \bar D^{ba}_\nu & \mathcal{F}^a_{h_{\mu\nu}} &= 0 \\ \mathcal{F}_{\rho\, a^a_\mu} &= \beta\sqrt{2}\kappa\, \bar g^{\mu\nu} \bar F^a_{\rho\nu} & \mathcal{F}_{\rho\, h_{\mu\nu}} &= \tfrac{\sqrt{2}}{\kappa} \left(\delta^{(\mu}_\rho \bar\nabla^{\nu)} - \frac{1+\alpha}{2} \bar g^{\mu\nu} \bar\nabla_\rho \right) . \end{aligned} \tag{7.41}$$

When we write down The Faddeev-Popov action

$$\begin{aligned}\tilde{S}[\bar{g}_{\mu,\nu}, \bar{A}^a_{\mu\nu}, h_{\mu\nu}, a^a_\mu, \bar{c}^a, \bar{c}^\mu, c^a, c^\mu] = S_{\mathrm{EYM}}[\bar{g}_{\mu\nu} + h_{\mu\nu}, \bar{A}^a_\mu + a^a_\mu] \\ + \int \mathrm{d}^d x \sqrt{\bar{g}} \Big(\tfrac{1}{2\xi_{\mathrm{YM}}} F^a F^a + \tfrac{1}{2\xi_{\mathrm{grav.}}} \bar{g}^{\mu\nu} F_\mu F_\nu \\ + g_{\mathrm{YM}} \bar{c}^a \mathcal{F}^a_{a^b_\mu} \theta^{b\, A^a_\mu} c^b + \tfrac{\kappa}{\sqrt{2}} \bar{c}^a \mathcal{F}^a_{a^b_\mu} \theta_\rho^{A^a_\mu} c^\rho + g_{\mathrm{YM}} \bar{c}^\rho \mathcal{F}_{\rho\, a^a_\mu} \theta^{b\, A^a_\mu} c^b + \tfrac{\kappa}{\sqrt{2}} \bar{c}^\rho \mathcal{F}_{\rho\, h_{\mu\nu}} \theta_\sigma^{g_{\mu\nu}} c^\sigma \Big), \end{aligned} \tag{7.42}$$

we have to take into account, that the measure of the d dimensional space-time integral $\mathrm{d}^d x \sqrt{g}$ depends on the background metric. Here, we also rescaled the Yang-Mills ghosts c^a by a factor of g_{YM} and the gravitational ghosts by $\frac{\kappa}{\sqrt{2}}$, in order to have the same canonical mass dimension for these as for the anti-ghost fields.

From equations (7.38) and (7.41) we can derive the ghost action

$$\begin{aligned} S_{\mathrm{gh}} = \int \mathrm{d}^d x \sqrt{\bar{g}} \Bigg(& \bar{c}^a \bar{g}^{\mu\nu} \bar{D}^{ab}_\mu D^{bc}_\nu c^c + \tfrac{\kappa}{\sqrt{2}} \bar{c}^a \bar{g}^{\mu\nu} \bar{D}^{ab}_\mu F^b_{\nu\rho} c^\rho + \beta\, g_{\mathrm{YM}} \kappa^2 \bar{c}^\mu \bar{g}^{\rho\sigma} \bar{F}^a_{\rho\mu} D^{ab}_\sigma c^b \\ & + \bar{c}^\mu \bar{g}^{\rho\sigma} \left(-\bar{\nabla}_\rho (g_{\mu\nu} \nabla_\sigma + g_{\sigma\nu} \nabla_\mu) + (1+\alpha) \bar{\nabla}_\mu g_{\rho\nu} \nabla_\sigma + \beta\, \kappa^2 \bar{F}^a_{\mu\rho} F^a_{\nu\sigma} \right) c^\nu \Bigg). \end{aligned} \tag{7.43}$$

We do not need to expand the full metric $g_{\rho\nu}$ and the covariant derivatives ∇_μ and D^{ab}_μ in the quantum fields $h_{\mu\nu}$ and a^a_μ. Instead, we can use equation (7.28) together with (7.39) to determine the free and interaction part of the ghost action:

$$\begin{aligned} S^0_{\mathrm{gh}} = \int \mathrm{d}^d x \sqrt{\bar{g}} \Bigg(& \bar{c}^a \bar{D}^{ab\,\mu} \bar{D}^{bc}_\mu c^c + \tfrac{\kappa}{\sqrt{2}} \bar{c}^a \bar{D}^{ab\,\mu} \bar{F}^b_{\mu\nu} c^\nu + \beta\, g_{\mathrm{YM}} \kappa^2 \bar{c}^\mu \bar{F}^a_{\rho\mu} \bar{D}^{ab\,\rho} c^b \\ & + \bar{c}^\mu \left(-\bar{g}_{\mu\nu} \bar{\nabla}^\rho \bar{\nabla}_\rho + \bar{R}_{\mu\nu} + \alpha\, \bar{\nabla}_\mu \bar{\nabla}_\nu + \beta\, \kappa^2 \bar{F}^{a\,\rho}_\mu \bar{F}^a_{\nu\rho} \right) c^\nu \Bigg) \end{aligned} \tag{7.44}$$

$$\begin{aligned} S^{\mathrm{int.}}_{\mathrm{gh}} = \int \mathrm{d}^d x \sqrt{\bar{g}} \Bigg(& \bar{c}^a \bar{D}^{ab}_\mu f^{bcd} a^{c\,\mu} c^d + \tfrac{\kappa}{\sqrt{2}} \bar{c}^a \bar{D}^{ac\,\mu} \left((\bar{D}^{bc}_\mu a^c_\rho) - (\bar{D}^{bc}_\rho a^c_\mu) + f^{bcd} a^c_\mu a^d_\rho \right) c^\rho \\ & - \beta\, g_{\mathrm{YM}} \kappa^2 \bar{c}^\mu \bar{F}^{a\,\rho}_\mu f^{abc} a^b_\rho c^b \\ & + \bar{c}^\mu \Big(-\bar{\nabla}^\rho (h_{\mu\nu} \bar{\nabla}_\rho + h_{\rho\nu} \bar{\nabla}_\mu + (\bar{\nabla}_\nu h_{\mu\rho})) + (1+\alpha) \bar{\nabla}_\mu (h_{\rho\nu} \bar{\nabla}^\rho + (\bar{\nabla}_\nu h_\rho{}^\rho)) \\ & + \beta\, \kappa^2 \bar{F}^{a\,\rho}_\mu ((\bar{D}^{ab}_\nu a^b_\rho) - (\bar{D}^{ab}_\rho a^b_\nu) + f^{abc} a^b_\nu a^c_\rho \Big) c^\nu \Bigg). \end{aligned} \tag{7.45}$$

7.3. Using Form in FRG computations

The functional renormalisation group in combination with background field theory is a powerful approach to quantum gravity calculations.

Our previous computations in a flat background required huge amount of straight forward, but tedious tensor calculus. The calculations in background field formalism with an arbitrary background metric $\bar{g}_{\mu\nu}$ will clearly be even more involved. Again, we decided to use the computer algebra system FORM. It can efficiently deal with large numbers of terms, which will occur in immediate steps of the calculations. The greatest advantage of FORM over other computer algebra systems are its powerful pattern matching capabilities. The pattern matcher of FORM can easily handle expressions with multiple free and contracted indices, which most other computer algebra systems fail to do. We developed some tools which allow for the automatisation of many tasks one encounters in the course of the calculation. The routines are not specific to functional renormalisation group calculation and can also be use for many other computations. The field expansion procedures 7.3.1 can be applied in generic background field theory calculations. In 7.3.2 we developed routines which allow to handle with derivative operators in FORM.

In contrast to the single run scripts we wrote for the one-loop calculations, e. g., section 4.5, we encoded the steps of the calculations in individual procedures which makes changes in the scripts much easier.

7.3.1. Field expansion

In our one-loop calculations, we limited ourselves to a flat background, i. e., $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$, $\bar{A}_\mu = 0$, and worked in momentum space for the beginning. For a flexible field expansion adoptable to arbitrary background configurations, it is preferable to start with the coordinate space representation of the action functional and deal with covariant derivatives which depend on the background fields. We defined two types of derivatives as functions: `D‘i’(mu,•)` and `Dym‘i’(mu,•)`. The first function is commutative and represents the metric conform covariant derivative, i. e., $\nabla_\mu(\bullet)$ with $\nabla_\mu(g_{\nu\rho}) = 0$. The second function is anti-commutative and represents the gauge covariant derivative in the adjoint representation, i. e., $D_\mu(\bullet) = \nabla_\mu(\bullet) + g_{\mathrm{YM}}[A_\mu, \bullet]$.

Above, we already introduced the counter `‘i’` which allows for multi-level background expansions. In addition to the usual background field expansion, e. g.,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \qquad A_\mu = \bar{A}_\mu + a_\mu$$

one might like to further expand the background fields

$$\bar{g}_{\mu\nu} = \bar{\bar{g}}_{\mu\nu} + \bar{h}_{\mu\nu}, \qquad \bar{A}_\mu = \bar{\bar{A}}_\mu + \bar{a}_\mu\,.$$

This is useful to calculate the functional derivative $\dfrac{\delta}{\delta \bar{g}_{\mu\nu}}$ with respect to the background fields. The use of the counter `‘i’` in all procedures makes it easy to iterate the expansion process. We denote the full fields and their functions by `i=0`, e. g., the full metric $g_{\mu\nu}$, Ricci scalar R are denoted by `g0(mu,nu)` and `R0`. The same index is used for the corresponding expansion field, e. g., `h0(mu,nu)`. We keep track of the order in the expansion field using the parameter `expC‘i’`, e. g., the graviton expansion of the metric is `g0(mu?,nu?) = g1(mu,nu)+ expC0*h0(mu,nu)`. The expansion parameter should be defined with a maximum power in order to automatically drop terms beyond our interest. For example, if we interested only in terms up to the quadratic order in the expansion fields

```
Autodeclare Symbol expC(:2);
```

will ensure that cubic terms are dropped in each expansion.

Clearly, the standard task is to expand all fields at once. This is done by the meta procedure `fullexpand`. Also, all expansion procedures should be called in the correct order, e. g., the expansion of the derivatives relies on the expansion of the Christoffel symbols.

```
#procedure fullexpand(i)


#call expandD(`i')
#call expandF(`i')
#call expandR(`i')
#call expandGamma(`i')
#call expandgI(`i')
#call expandSqrtg(`i')

*expand the metric
id g`i'(mu?,nu?) = g{`i'+1}(mu,nu)+ expC`i'*h`i'(mu,nu);
*expand (background covariant) derivatives of the metric
* using metric conformity of the background covariant derivative
id D{`i'+1}(mu1?,g`i'(mu?,nu?)) = expC`i'*D{`i'+1}(mu1,h`i'(mu,nu));

#call unwrap({`i'+1})

#endprocedure
```

The call of `unwrap({`i'+1})` simplifies the background covariant derivatives. We will define the procedure in section 7.3.2.

The first step is the expansion of the derivatives[1]. The metric dependence of the derivatives is deduced form the number of space-time indices of the argument of the derivative. We add the change of the connection `Gamma`i'` which carries the dependence on the expansion field `h`i'(mu,nu)`, e. g., the expansion of the derivative of a covariant vector field $\mathcal{V}_\mu$ is

$$\nabla_\mu(\mathcal{V}_\nu) = \bar{\nabla}_\mu(\mathcal{V}_\nu) - \hat{\Gamma}^\rho_{\mu\nu}\mathcal{V}_\rho\,.$$

The definition of $\hat{\Gamma}^\rho_{\mu\nu}$ will be given below, when we explain the expansion of the corresponding FORM function `Gamma`i'`. The procedure also takes caution of the correct sign for contravariant vector fields, e. g., the gravitational Faddeev-Popov ghosts. The charged fields, which we encounter in our computations, are in the adjoint representation and in all terms we trace over the gauge indices. Thus, the expenasion of the gauge covariant derivative of a charged scalar field $\mathcal{S}$ becomes:

$$D_\mu(\mathcal{S}) = \bar{D}_\mu(\mathcal{S}) - i[a_\mu, \mathcal{S}]$$

To allow for a simple implementation in FORM, we defined the fields and functions thereof, e. g., fieldstrengh tensors `F`i'(mu,nu)`, as non-commuting tensors.

```
#procedure expandD(i)

*scalars
id D`i'(mu?,t?)=D{`i'+1}(mu,t);

*scalars (charged fields);
id Dym`i'(mu?,t?)=Dym{`i'+1}(mu,t)-i_*expC`i'*(a`i'(mu)*t-t*a`i'(mu));
```

[1]The full script contains rules for second derivatives $\nabla_\mu\nabla_\nu$ and $D_\mu D_\nu$ as well.

```
*vectors (covariant)
repeat;
id,once,D`i'(mu?,t?!contra(nu?))=D{`i'+1}(mu,t(nu))
                                 -Gamma`i'(intI,mu,nu)*t(intI);
sum intI;
endrepeat;

*vectors (covariant) (charged fields)
repeat;
id,once,Dym`i'(mu?,t?!contra(nu?))=Dym{`i'+1}(mu,t(nu))
                                   -Gamma`i'(intI,mu,nu)*t(intI)
                            -i_*expC`i'*(a`i'(mu)*t(nu)-t(nu)*a`i'(mu));
sum intI;
endrepeat;

*vectors (contra-variant)
repeat;
id,once,D`i'(mu?,t?contra(nu?))=D{`i'+1}(mu,t(nu))
                                +Gamma`i'(nu,mu,intI)*t(intI);
sum intI;
endrepeat;

*vectors (contra-variant) (charged fields)
repeat;
id,once,Dym`i'(mu?,t?contra(nu?))=Dym{`i'+1}(mu,t(nu))
                                  +Gamma`i'(nu,intI,mu)*t(intI)
                            -i_*expC`i'*(a`i'(mu)*t(nu)-t(nu)*a`i'(mu));
sum intI;
endrepeat;


*tensors (covariant)
repeat;
id,once,D`i'(mu?,t?(nu1?,nu2?))=D{`i'+1}(mu,t(nu1,nu2))
                                -Gamma`i'(intI,mu,nu1)*t(intI,nu2)
                                -Gamma`i'(intI,mu,nu2)*t(nu1,intI);
sum intI;
endrepeat;

*tensors (covariant)  (charged fields)
repeat;
id,once,Dym`i'(mu?,t?(nu1?,nu2?))=Dym{`i'+1}(mu,t(nu1,nu2))
                                  -Gamma`i'(intI,mu,nu1)*t(intI,nu2)
                                  -Gamma`i'(intI,mu,nu2)*t(nu1,intI)
                  -i_*expC`i'*(a`i'(mu)*t(nu1,nu2)-t(nu1,nu2)*a`i'(mu));
sum intI;
endrepeat;
```

```
#endprocedure
```

Next, we expand the curvature quantities, i. e., the Ricci scalar and tensor, as well as the Riemann tensor:

$$R^\rho{}_{\sigma\mu\nu} = \bar{R}^\rho_{\sigma\mu\nu} - \bar{\nabla}_\mu \left(\hat{\Gamma}^\rho_{\sigma\nu}\right) + \bar{\nabla}_\nu \left(\hat{\Gamma}^\rho_{\sigma\mu}\right) + \hat{\Gamma}^\alpha_{\mu\sigma}\hat{\Gamma}^\rho_{\nu\alpha} - \hat{\Gamma}^\alpha_{\nu\sigma}\hat{\Gamma}^\rho_{\mu\alpha}$$

```
#procedure expandR(i)

*write Ricci scalar as contraced Ricci tensor
repeat;
id,once,R`i' = gI`i'(intI1,intI2)*R`i'(intI1,intI2);
sum intI1,intI2;
endrepeat;

*write Ricci tensor as contraced Riemann tensor
repeat;
id,once,R`i'(mu1?,mu2?)= R`i'(intI,mu1,intI,mu2);
sum intI;
endrepeat;

repeat;
id,once,R`i'(mu1?,mu2?,mu3?,mu4?)= R{`i'+1}(mu1,mu2,mu3,mu4)
                                   - D{`i'+1}(mu3,Gamma`i'(mu1,mu2,mu4))
                                   + D{`i'+1}(mu4,Gamma`i'(mu1,mu2,mu3))
                         + Gamma`i'(intI,mu3,mu2)*Gamma`i'(mu1,mu4,intI)
                         - Gamma`i'(intI,mu4,mu2)*Gamma`i'(mu1,mu3,intI) ;
sum intI;
endrepeat;

#endprocedure
```

The expansion of the field strength tensor:

$$F_{\mu\nu} = \bar{F}_{\mu\nu} + \bar{D}_\mu\left(a_\nu\right) - \bar{D}_\nu\left(a_\mu\right) - i[a_\mu, a_\nu]$$

is implemented in the procedure `expandF`

```
#procedure expandF(i)
id F`i'(mu?,nu?)=F{`i'+1}(mu,nu)
            +expC`i'*Dym{`i'+1}(mu,a`i'(nu))
            -expC`i'*Dym{`i'+1}(nu,a`i'(mu))
            -i_*expC`i'*expC`i'*(a`i'(mu)*a`i'(nu)-a`i'(nu)*a`i'(mu));
id Dym{`i'+1}(rho?,F`i'(mu?,nu?))=Dym{`i'+1}(rho,F{`i'+1}(mu,nu)
            +expC`i'*Dym{`i'+1}(mu,a`i'(nu))
            -expC`i'*Dym{`i'+1}(nu,a`i'(mu))
            -i_*expC`i'*expC`i'*(a`i'(mu)*a`i'(nu)-a`i'(nu)*a`i'(mu)));

#endprocedure
```

In the expansion of the covariant derivatives ∇_μ and curvature quantities we used the function `Gamma‘i’`. It is not the full Christoffel symbol, but rather the difference between the connection of the full covariant derivative ∇_μ and of the background covariant derivative $\bar{\nabla}_\mu$

$$\nabla_\mu (\mathcal{V}_\nu) = \bar{\nabla}_\mu (\mathcal{V}_\nu) - \hat{\Gamma}^\rho_{\mu\nu} \mathcal{V}_\rho.$$

It can be express in terms of the full inverse metric and the graviton field as

$$\hat{\Gamma}^\rho_{\mu\nu} = \tfrac{1}{2} g^{\rho\sigma} (\bar{\nabla}_\mu h_{\nu\sigma} + \bar{\nabla}_\nu h_{\mu\sigma} - \bar{\nabla}_\sigma h_{\mu\nu}).$$

```
#procedure expandGamma(i)

*standard
repeat;
id,once,Gamma‘i’(mu?,nu1?,nu2?)=1/2*expC‘i’*gI‘i’(mu,intI)*(
                                               D{‘i’+1}(nu1,h‘i’(nu2,intI))
                                              +D{‘i’+1}(nu2,h‘i’(nu1,intI))
                                              -D{‘i’+1}(intI,h‘i’(nu1,nu2)) );
sum intI;
endrepeat;

*in derivative
repeat;
id,once,D{‘i’+1}(nu?,Gamma‘i’(mu?,nu1?,nu2?))=1/2*expC‘i’
                                    *D{‘i’+1}(nu,gI‘i’(mu,intI)*(
                                          D{‘i’+1}(nu1,h‘i’(nu2,intI))
                                         +D{‘i’+1}(nu2,h‘i’(nu1,intI))
                                         -D{‘i’+1}(intI,h‘i’(nu1,nu2)) ));
sum intI;
endrepeat;
#call unwrap({‘i’+1})

#endprocedure
```

Finally, we expand the inverse metric $g^{\mu\nu}$ and the density factor $\sqrt{g}$. Until now, all expansions are exact; $g^{\mu\nu}$ and $\sqrt{g}$ are the only quantities whose expansion does not terminate. In this example we give only the expansion up the second order to maintain readability:

```
*inverse metric
#procedure expandgI(i)

*standard
repeat;
id,once,gI‘i’(mu?,nu?)= gI{‘i’+1}(mu,nu)
         - expC‘i’*gI{‘i’+1}(mu,intI1)
                       *h‘i’(intI1,intI2)*gI{‘i’+1}(intI2,nu)
         + expC‘i’^2*gI{‘i’+1}(mu,intI1)*h‘i’(intI1,intI2)
           *gI{‘i’+1}(intI2,intI3)
```

```
                        *h`i'(intI3,intI4)*gI{`i'+1}(intI4,nu);
sum intI1,...,intI4;
endrepeat;

*in derivatives
repeat;
id,once,D{`i'+1}(nu1?,gI`i'(mu?,nu?))=
    -expC`i'*gI{`i'+1}(mu,intI1)*D{`i'+1}(nu1,h`i'(intI1,intI2))
                                        *gI{`i'+1}(intI2,nu)
    +expC`i'^2*gI{`i'+1}(mu,intI1)*gI{`i'+1}(intI2,intI3)
     *D{`i'+1}(nu1,h`i'(intI1,intI2)*h`i'(intI3,intI4))
     *gI{`i'+1}(intI4,nu);
sum intI1,...,intI4;
endrepeat;

#endprocedure

*square root of g
#procedure expandSqrtg(i)
repeat;
id,once,sqrtg`i' = sqrtg{`i'+1}*(1
      + 1/2*expC`i'*gI{`i'+1}(intI1,intI2)*h`i'(intI1,intI2)
      +  1/8*expC`i'^2*gI{`i'+1}(intI1,intI2)*gI{`i'+1}(intI3,intI4)*(
                                   h`i'(intI1,intI2)*h`i'(intI3,intI4)
                                -2*h`i'(intI1,intI3)*h`i'(intI2,intI4));
sum intI1,...,intI4;
endrepeat;
#endprocedure
```

The expansion to higher orders is straight forward.

7.3.2. Derivative manipulations

FORM's dictionary contains many useful types of variables like vectors, tensors, and even Dirac gamma matrices, but no special objects with the properties of derivatives. Therefore, we need to define our own routines to handle derivative operators. As we introduced above, in our notation covariant derivatives are functions whose name starts with a capital `D`, followed by `ym` in case it is non-commutative, and ends with a number indicating the expansion level. Its first argument is a space-time index; the second is the terms the derivative acts on, e. g., the background covariant derivative acting on the full Ricci tensor would be written as

$$\bar{\nabla}_\mu(R_{\nu\rho}) = \texttt{D1(mu,R0(nu,rho))}\,.$$

In addition to the single derivatives, we define functions for double derivatives. These have either three arguments (two space-time indices and one term) or one argument (one term and no indices) in case of the Laplacian.

We will often encounter terms in which the gauge covariant derivative `Dym` acting on uncharged fields. In this case we can substitute it by `D`

```
#procedure renameDsNeutrals(i)
```

```
* rename derivatives of uncharged objects
id Dym`i'(mu?,t?!gauge) = D`i'(mu,t);
id Dym`i'(mu?,t?!gauge(?y)) = D`i'(mu,t(?y));
id Dym`i'(mu?,t?!gauge(rho1?,rho2?)) = D`i'(mu,t(rho1,rho2));
id DDym`i'(?x,t?!gauge(?y)) = DD`i'(?x,t(?y));
id DDym`i'(?x,t?!gauge) = DD`i'(?x,t);
id DDym`i'(?x,t?!gauge(rho1?,rho2?)) = DD`i'(?x,t(rho1,rho2));
#endprocedure
```

We also define a clean-up procedure which simply sets derivatives of constant terms to zero:

```
#procedure DMcleanup(i)

* symbols and numbers are always constants
id D`i'(mu?,x?number_)=0;
id D`i'(mu?,x?symbol_)=0;

* rewrite 2nd derivatives
id D`i'(mu?,D`i'(?x))=DD`i'(mu,?x);

* metric conformity
id D`i'(mu?,g`i'(nu1?,nu2?))=0;
id D`i'(mu?,gI`i'(nu1?,nu2?))=0;

* remove derivatives of constants
id D`i'(mu?,t?!nc`i') = 0;
id D`i'(mu?,t?!nc`i'(mu?)) = 0;
id D`i'(mu?,t?!nc`i'(mu?,nu?)) = 0;

#endprocedure
```

We use variables of the type `symbol` only for constants, e. g., coupling constants or the expansion parameter. Their derivatives are removed be the statement

```
id D`i'(mu?,x?symbol_)=0;
```

To identify non-constant tensors, we use the sets `` nc`i' ``. These contain all tensors which are not constants with respect to the derivative `` D`i' ``. Now, the statements

```
id D`i'(mu?,t?!nc`i') = 0;
id D`i'(mu?,t?!nc`i'(mu?)) = 0;
id D`i'(mu?,t?!nc`i'(nu?,rho?)) = 0;
```

set all derivatives of tensors which are not in `` nc`i' `` to zero.

There is no need for a separate clean-up function for gauge covariant derivatives; one can simply include the tag `ym` in the procedure argument, e. g.,

```
#call DMcleanup(ym1)
```

will remove derivatives of constants for `Dym1` the background covariant derivative $\bar{D}_\mu$.

Unwrap

When we simplify expressions which involve derivatives, the most common task is to rewrite derivatives of sums and products into derivatives of the fields, e. g.,

$$\bar{\nabla}_\mu \left(\bar{g}^{\mu\rho} + h^{\mu\rho} + h^{\mu\alpha} h_\alpha{}^\rho\right) = \bar{\nabla}_\mu \left(h^{\mu\rho}\right) + \bar{\nabla}_\mu \left(h^{\mu\alpha}\right) h_\alpha{}^\rho + h^{\mu\alpha} \bar{\nabla}_\mu \left(h_\alpha{}^\rho\right)$$

`unwrap` procedure is a meta procedure applies the sum rule and the Leibniz rule for both, standard and gauge covariant derivatives:

```
#procedure unwrap(i)

*make all derivatives non-commuting functions
id D‘i’(?x)=Dym‘i’(?x);
id DD‘i’(?x)=DDym‘i’(?x);
argument Dym‘i’,2;
id D‘i’(?x)=Dym‘i’(?x);
id DD‘i’(?x)=DDym‘i’(?x);
endargument;

*pull symbols and numbers out of the derivatives
normalize,Dym‘i’;

#call DMsumrule(ym‘i’)

#call DMleibniz(ym‘i’)

#call DMcleanup(ym‘i’)

#call renameDsNeutrals(‘i’)
* rename derivatives of uncharged objects

#endprocedure
```

In the first step we change all commuting `D`'s into non-commuting `Dym`'s. This allows us deal with both types of derivatives at once. The change is reverted in the end by the call of `renameDsNeutrals`. The sum rule can be implemented straight forwardly using the FORM statement `splitarg`:

```
#procedure DMsumrule(i)
splitarg D‘i’;
repeat id D‘i’(mu?,?x,y?,z?) = D‘i’(mu,?x,y)+D‘i’(mu,z);
#endprocedure
```

`splitarg D‘i’;` puts all terms of a sum in one argument of `D‘i’` into a separate argument. In the next step, we only need to split the multiple argument “derivative” into a sum of derivatives.

The implementation of the Leibniz rule is more elaborate:

```
#procedure DMleibniz(i)
argument D‘i’;
```

```
#call DMputinF()
endargument;
repeat;
normalize,D‘i’;
id D‘i’(mu?,f(x?,?y))=D‘i’(mu,x)*f(?y)+x*D‘i’(mu,f(?y));
chainout f;
id f(x?)=x;
id D‘i’(mu?,f)=0;
endrepeat;
#endprocedure
```

Form features no statement similar to `splitarg` doing the same trick for products in function arguments. However, we can use an auxiliary function `f` and the `chainin` statement to bring products into a form we can manipulate further. The procedure `DMputinF`

```
#procedure DMputinF()
*put factors in f
repeat id f1?!{,f}(?x)=f(f1(?x));
repeat id t?(?x)=f(t(?x));
repeat id t?(mu?,nu?)=f(t(mu,nu));
repeat id x?=f(x);
chainin f;
#endprocedure
```

puts all factors of a product into separate arguments of the auxiliary function `f`, e. g.,

$$\texttt{a0(mu)*a0(nu)} \to \texttt{f(a0(mu),a0(nu))}\,.$$

The product rule can now be executed by the Form statement

```
id D‘i’(mu?,f(x?,?y))=D‘i’(mu,x)*f(?y)+x*D‘i’(mu,f(?y));
```

We finally remove the auxiliary function using `chainout`:

```
chainout f;
id f(x?)=x;
id D‘i’(mu?,f)=0;
```

Partial Integration

The expansion will yield terms which are identical up to total derivative. To collect these terms we have to partially integrate some of them.

By partial integration we can shuffle covariant derivatives between factors as

$$\int \sqrt{g}\mathcal{A}^\mu\nabla_\mu(\mathcal{B}) = -\int \sqrt{g}\nabla_\mu(\mathcal{A}^\mu)\mathcal{B} + \text{boundary terms} \tag{7.46}$$

in the action functional with generic terms $\mathcal{A}^\mu$ and $\mathcal{B}$. The same formula applies to the gauge covariant derivatives D_μ.

To implement partial integration we use an auxiliary function `f` to rewrite products; the same way we did before for the Leibniz rule:

```
#procedure pInt(i)

*make all derivatives non-commuting functions
id D`i'(?x)=Dym`i'(?x);
id DD`i'(?x)=DDym`i'(?x);

#call DMputinF()

id f(?x,Dym`i'(mu?,?y),?z)=-Dym`i'(mu,f(?x))*f(?y,?z)
                           -f(?x,?y)*Dym`i'(mu,f(?z));
id Dym`i'(mu?,f)=0;
repeat;
id Dym`i'(mu?,f(x?,?y))=Dym`i'(mu,x)*f(?y)+x*Dym`i'(mu,f(?y));
chainout f;
id f(x?)=x;
id Dym`i'(mu?,f)=0;
endrepeat;

#call DMcleanup(ym`i')

#call renameDsNeutrals(`i')
* rename derivatives of uncharged objects

*rewrite Laplace
id DDym`i'(mu?,nu?,?x)*gI`i'(mu?,nu?)=DDym`i'(?x);
id DD`i'(mu?,nu?,?x)*gI`i'(mu?,nu?)=DD`i'(?x);

#endprocedure
```

This procedure would naively pick up terms $\sim \nabla_\mu(\sqrt{g})$ which are absent in (7.46). We circumvent this problem by defining the density factors $\sqrt{g}$, $\sqrt{\bar{g}}$ etc. as constants with respect to the covariant derivative. Thus these terms are removed by `DMcleanup`.

Partial integration is also useful when we want to reduce the number of terms by imposing the gauge fixing condition for Landau type gauges, i. e., when we set the corresponding gauge parameter to zero, e. g., $\bar{g}^{\mu\nu}\bar{D}_\mu(a_\nu) = 0$ for the gluons. In this case terms like

$$\sqrt{\bar{g}}\bar{g}^{\mu\nu}\bar{D}_\mu(\dots)a_\nu \tag{7.47}$$

vanish as well. If we want to exploit this, the easiest way is to first partially integrate such terms and then impose the gauge condition. In our FORM scripts this looks as follows:

```
if (match(Dym1(mu?,?x)*a0(nu?)*gI1(mu?,nu?)));
#call pInt(1)
#call unwrap(1)
endif;
```

When we finally impose the gauge fixing

```
id Dym1(mu?,a0(nu?))*gI1(mu?,nu?)=0;
```

it will remove terms of the type (7.47) as well.

8. Interpretation of the Gravitational Corrections to Running Couplings

The work of Robinson and Wilczek [11] started a series of studies of the gravitational corrections to running of gauge couplings. We like to give an overview of the literature and comment on some of the various results. Concludingly, we present our own point of view.

In his first contribution [13] to the discussion Toms used dimensional regularisation and found that the gravitational corrections to gauge couplings vanish. This result is rather trivial, since dimensional regularisation is insensitive to quadratic divergences which are responsible to the non-zero results observed by Robinson and Wilczek.

The inclusion of a finite cosmological constant in his subsequent calculation [14] lead to a non-vanishing contribution of the running of gauge couplings. The dimensionless combination $\kappa^2\Lambda$ of the gravitational coupling and cosmological constant appears in front of the logarithmic divergences leading to the running of the coupling. This observation motivated the use of dimensional regularisation in our analysis of the Einstein-Yukawa system [32]. Here, the masses of the scalar and fermion fields combine with the gravitational coupling to dimensionless expressions of the form $\kappa^2 m^2$.

Later, Toms revised his computations [22], still using the Vilkovisky-DeWitt background field method. Instead of dimensional regularisation, he now used a proper time cut-off in the heat kernel representation of the one-loop corrections. He claimed, that his result of a non-zero gravitational correction to the gauge coupling is independent of the chosen gauge fixing condition, since it was computed in the Vilkovisky-DeWitt formalism. This conclusion was challenged by Nielsen [24] who showed that the gauge fixing independence of the Vilkovisky-DeWitt formalism holds only for logarithmically, but not for the quadratically divergent terms.

The author and collaborators [15, 16] used Feynman graph techniques obtain the gravitational one-loop corrections. We found that in de Donder gauge all gravitational contributions to the running of the Yang-Mills coupling cancel in cut-off regularisation.

In their first work on the gravitational contribution of the running of gauge couplings [18], Tang and Wu used the so called loop-regularisation and found a non-zero gravitational correction to the Yang-Mills β function. They later combined their regularisation scheme with the Vilkovisky-DeWitt framework in two articles with contradicting results [19, 20]. The result of the latter of the articles agrees with the result of He, Wang and Xianyu [23] who employed a similar regularisation scheme.

A non-perturbative analysis of the renormalisation group flow of the couplings of the Einstein-Yang-Mills system in the framework of functional renormalisation group was done by Daum, Harst and Reuter [21]. They found an non-zero gravitational contribution to the β function of the gauge coupling.

In a further investigation using functional renormalisation group methods, Folkerts, Litim and Pawlowski [25] worked out the regulator and gauge fixing dependence of the gravitational contributions to the Yang-Mills β function. They showed that the one-loop contributions are negative, hence acting in the direction of asymptotic freedom, for all regulator functions and a gravitational gauge fixing parameter ≥ -1.

In [26], Felipe, Brito, Sampaio and Nemes formulated the gravitational corrections to quantum electrodynamics in terms of a surface term form the momentum integration, thereby underlining the scheme dependence of the various result in the literature.

In 2010, two articles discussed the physical interpretation of a possible quadratic energy dependence of running couplings. The first were Anber, Donoghue and El-Houssieny [27] who analysed the gravitational corrections in scattering processes with Yukawa and ϕ^4 interactions. They defined the renormalised coupling constant from the *full* gravitational one-loop corrections, including those terms we used to renormalise higher derivative operators. They showed that in contrast to logarithmic corrections to the coupling constants, quadratic corrections cannot be defined consistently from kinematic quantities, i. e., depending on whether one uses the s- or the t-channel to identify the renormalisation scale, one gets a different sign for the β function. This criticism does not apply to our line of thought. We did not include higher derivative corrections in our definition of the running coupling, hence the β function does not depend on the external momenta and the momentum transfer.

Furthermore, they argued that the cut-off dependence of the coupling constants should not be understood as the running of the coupling constant. The observed physical coupling is independent of the arbitrary cut-off scale. Thus, in their definition which is motivated from scattering amplitudes, this dependence should not be interpreted as a running of the coupling constant. We will comment on this later.

In reaction to the criticism of [27], Toms [30] revised his calculations concluding that quadratic divergences should not be viewed as leading to a running of the charge.

Ellis and Mavromatos [28, 29] connected the quadratic energy dependence of the β function to a higher-derivative term of the form $F_{\mu\nu}D^2F^{\mu\nu}$ whose coefficient can be changed by a local field redefinition without changing the S-matrix. The identification of the quadratic energy dependence with such a higher derivative term is not correct as it assumes a transmutation of the cut-off scale dependence to a momentum dependence. The momentum dependence of the scattering amplitudes is related to the higher derivative terms whose renormalisation is independent of the quadratic divergences, see also Toms' reply to [29].

Our conclusion

As a reaction to the various opinions in the literature, we now like to present our point of view on the topic of gravitional corrections to the running of coupling constants. Our definition of the renormalisation scale (chapter 4) does not rely on the kinematics of scattering processes. We rather introduced it as a sliding scale [54]. The β function encodes the change of the coupling parameters when the energy cut-off is varied. A power-law momentum dependence of scattering matrix elements, on the other hand, does not correspond to a running coupling as we defined it. Let us illustrate this by assuming that the gravitational contribution to the β function is non-zero, i. e.,

$$\beta_g = ag^3 + bg\kappa^2\mu^2 \,. \tag{8.1}$$

The first term is the usual one-loop β function in absence of gravity, with a depending of the matter content of the theory. The second term is the gravity induced running of the coupling constant. In addition to the renormalisation of g, we know that gravitational corrections will also renormalise higher derivative operators of the fields, see chapter 3. The gauge theory part (i. e. we ignore the purely gravitational interaction) of a matrix element

$\mathcal{M}$ for fermion–fermion scattering in Einstein-Yang-Mills theory would have the form[1]

$$\mathcal{M} \sim g^2(1 + A\log(-q^2) + Bq^2 + Cq^2\log(-q^2) + \cdots) . \tag{8.2}$$

The first quantum correction $A \sim g^2$ arises from the non-gravitational one-loop contributions and $C \sim \kappa^2$ corresponds to the gravitational corrections to the higher derivative operators. These terms can be read off the β functions of g and of the parameters of the higher derivative operators, respectively. The $\log(-q^2)$ corrections are usually attributed to the running of the coupling constant, which is tantalising, but entirely true. The logarithmic divergences are related to non-local terms $\sim \log(\nabla^2)$ in the effective action. A and C originate from these non-local terms in the effective action and not directly from the dependence of the coupling constant on the artificial energy scale μ.

The confusion in the literature is about the origin of Bq^2 and its interpretation. Is B related to the possible quadratically energy depending part of the β function of g? The answer is *negative*. The parameter B is solely determined by the coefficients of the higher derivative terms. We know that these terms have to be included in the action of the effective field theory, since the gravitational one-loop divergences require counterterms of this form, but neither the bare nor renormalised values of the coefficients can be determined form our renormalisation group analysis. A non-perturbative fix point analysis of the renormalisation group behaviour could constrain these coefficients [2], but from our point of view of an effective field theory these coefficients are free parameters and have to be determined from experimental data.

The gravitational corrections to the β function, i. e., b in equation (8.1), on the other hand will not show up in scattering amplitudes. In this point, we agree with Anber, Donoghue and El-Houssieny [27]. A direct physical, i. e., observable effect of the quadratically energy dependent part of the β is already ruled out by several proofs [12, 24, 25] of its gauge and regulator dependence. Still, such corrections cannot be simply ignored. One should bear in mind that the coupling parameters in the action, i. e., g, λ, $\mathcal{Y}$ and κ, are not identical to the physical charges. The relationship between the coupling parameters and physical charges always depends on the calculation scheme, i. e., regularisation and renormalisation scheme and at least for theories that include gravity also the gauge fixing. A renormalised quantum field theory always depends on the renormalisation scale explicitly, via the regularisation prescription, and implicitly, via the β functions and anomalous dimensions. Since the explicit scale dependences vary with the calculation scheme, even a scheme dependent β cannot be dropped. Otherwise, the computation of observables, which have to be independent of the renormalisation scale, is not possible.

In conclusion, the question is simply what one likes to call "running coupling". Anber *et al.* include the contributions of non-local terms in their definition of the physical charge, hence are interested in an observable variation of the charge when changing the energy. Our point of view is a puristic Wilsonian one: The coupling constants depend on the scale at which the theory is renormalised, in order to yield correct predictions for physical observables which are independent of the renormalisation scale.

[1] Compare with equation (1) in [27].

[2] This question was discussed for higher derivative gravity in [65].

9. Summary and Conclusion

We analysed the influence of quantum gravitational effects on the renormalisation of the Standard Model fields and interactions. The framework we worked in was a large extra dimensions scenario with a freely moving matter brane. The pure gravitational one-loop effects of brane displacements, i. e., those which do not depend on the brane tension, were found to cancel for all matter fields and independently of the number of brane and bulk dimensions.

In chapter 4, we have established a cut-off regularisation applicable in extra-dimensional scenarios which involves a cut-off of the $(d+\delta)$-dimensional momentum and a fixed parametrisation of the loops, where the graviton propagator does not carry any external momenta. This particular parametrisation is completely fixed by the demand of gauge invariance of the counterterms and by requiring that all bubbles, triangles, etc. are parametrised in identical manner.

Using this regularisation, we determined the gravitational one-loop contributions of the β functions of Yang-Mills gauge coupling, quartic scalar self-interaction and Yukawa coupling. We found that for the physical 3-brane ($d=4$) the possible gravitational corrections to the running gauge coupling cancels

$$\beta_g\Big|_{\mathcal{O}(\kappa^2)}^{D=d+\delta} = -\frac{g}{(4\pi)^{D/2-1}\Gamma(\frac{D}{2})}\frac{d-4}{D-2}\left((d-3)(d-2)+\frac{\delta}{d}(d^2-4d+8)\right)\frac{\mu^{D-2}}{M_{(D)}^{D-2}}.$$

The leading gravitational corrections to the quartic scalar self-interaction and Yukawa coupling are both positive in $4+\delta$ dimensions

$$\beta_\lambda\Big|_{\mathcal{O}(\kappa^2)}^{D=4+\delta} = \frac{\lambda}{(4\pi)^{1+\delta/2}\Gamma\left(\frac{\delta}{2}+2\right)M_{(D)}^{\delta+2}}\left(\frac{12(\delta+4)}{\delta+2}\mu^{2+\delta}+\frac{2\left(\delta^2+50\delta-32\right)}{\delta+2}m_\varphi^2\mu^\delta\right),$$

$$\beta_{\mathcal{Y}}\Big|_{\mathcal{O}(\kappa^2)}^{D=4+\delta} = \frac{\mathcal{Y}}{(4\pi)^{1+\delta/2}\Gamma\left(\frac{\delta}{2}+2\right)M_{(D)}^{\delta+2}}\left(\frac{15}{2}\mu^{2+\delta} + \left\{\frac{\delta^2+26\delta+16}{2(\delta+2)}m_\varphi^2-\frac{19\delta^2-198\delta+128}{8(\delta+2)}m_\psi^2\right\}\mu^\delta\right).$$

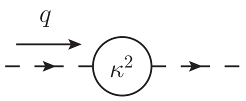
The gravitational one-loop corrections to the β functions are known to depend on the gauge fixing condition and regularisation scheme. Thus, our results only apply to calculations in de Donder gauge and with a sharp momentum cut-off.

Furthermore, we determined the higher derivative counterterms for scalar, fermionic and gauge fields. The counterterms involving two fermions or two scalars are a linear combination of all possible operators (5.12), (5.7) and not only the Lee-Wick terms, as in the gauge sector (5.15). This contradicts the results of [45] and leads to the conclusion that there is no connection between the gravitational one-loop counterterms of standard model matter and the Lee-Wick standard model. It is important to note that the appearance of the Lee-Wick term for gauge fields as the only gauge field counterterm, although it is tempting, does not allow to conclude the existence of the massive particle associated with the Lee-Wick term in the Lee-Wick standard model, as has been done by [44]. In the context considered here, the Lee-Wick term is only one term in an infinite series of counterterms of a non-renormalisable field theory. One might argue that the higher order terms are exactly such that they correspond to further new particles, as in the higher derivative Lee-Wick standard model proposed by Carone and Lebed [66]. However, it is in general not possible to conclude the existence of further particles from the appearance of special counterterms in an effective field theory, because these terms are only the residual low-energy effects of the unknown physics at high energies.

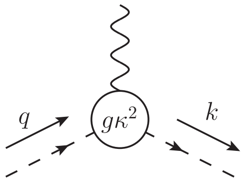
A. Field Renormalisation in $d + \delta$ Dimensions

In chapter 5 we gave only the results for the most interesting scenario with $D = 4 + \delta$ space-time dimensions. Our actual computations were done for general $(d-1)$ brane, i. e., for field in d dimensions and δ extra dimensions.

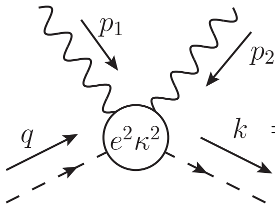
A.1. Scalars



$$
\begin{aligned}
= \frac{i}{(4\pi)^{D/2-1}\Gamma\left(\frac{D}{2}\right) M_{(D)}^{D-2}} \Bigg[& m_\phi^2 \left(\frac{2d(d(D-1)-2)}{D-2} \frac{\Lambda^{D-2}-\mu^{D-2}}{D-2} \right. \\
& \left. + \frac{4d(\delta-2)}{d-2} m_\phi^2 \frac{\Lambda^{D-4}-\mu^{D-4}}{D-4} \right) \\
& - q^2 \left(\frac{2(d-2)(d^2(D-3)+\delta(2-d))}{d(D-2)} \frac{\Lambda^{D-2}-\mu^{D-2}}{D-2} \right. \\
& \left. - \frac{4d((2d(3-\delta)+11\delta)-20)-40\delta+64}{d(d-2)} m_\phi^2 \frac{\Lambda^{D-4}-\mu^{D-4}}{D-4} \right) \\
& + q^2 \frac{4\delta(d-2)(d-1)}{d(d+2)} \frac{\Lambda^{D-4}-\mu^{D-4}}{D-4} \Bigg] + \cdots \qquad \text{(A.1)}
\end{aligned}
$$



$$
\begin{aligned}
&= \frac{i\, g t^a}{(4\pi)^{D/2-1}\Gamma\left(\frac{D}{2}\right) M_{(D)}^{D-2}} \Bigg[-(q+k)^\mu \Bigg(\frac{2(d-2)}{d(D-2)} (d^2(D-3) \\
&\qquad + \delta(2-d)) \frac{\Lambda^{D-2}-\mu^{D-2}}{D-2} \\
&\qquad + \frac{8(d-2)(\delta-2)}{d} m_\phi^2 \frac{\Lambda^{D-4}-\mu^{D-4}}{D-4} \Bigg) \\
&\qquad + \Big\{ (q^2+k^2) q^\mu + (k^2+q^2) k^\mu \Big\} \frac{4\delta(d-2)(d-1)}{d(d+2)} \frac{\Lambda^{D-4}-\mu^{D-4}}{D-4} \\
&\qquad + \Big\{ (k^2-qk) q^\mu + (q^2-kq) k^\mu \Big\} \frac{4}{d\,(d^2-4)} (96 - (124\delta \\
&\qquad + d(88 - 28\delta + d(8 - 25\delta + 2d(-11 + 2D + \delta))))) \frac{\Lambda^{D-4}-\mu^{D-4}}{D-4} \Bigg] + \cdots
\end{aligned}
\tag{A.2}
$$



$$
\begin{aligned}
&= \frac{i\, e^2}{(4\pi)^{D/2-1}\Gamma\left(\frac{D}{2}\right) M_{(D)}^{D-2}} \Bigg[2\eta^{\mu\nu} \Bigg(\frac{2(d-2)}{d(D-2)} (d^2(D-3) \\
&\qquad + \delta(2-d)) \frac{\Lambda^{D-2}-\mu^{D-2}}{D-2} \\
&\qquad + \frac{8(d-2)(\delta-2)}{d} m_\phi^2 \frac{\Lambda^{D-4}-\mu^{D-4}}{D-4} \Bigg) \\
&\qquad + \Big\{ 2\big(\eta^{\mu\nu}(q^2+k^2) - p_1^\nu p_2^\mu + (q+k)^\mu (q+k)^\nu\big) \Delta_1 \\
&\qquad + \big(\eta^{\mu\nu}(p_1+p_2)\cdot(p_1+p_2) - (p_1+p_2)^\mu (p_1+p_2)^\nu\big) \Delta_2 \\
&\qquad + 4(p_1\cdot p_2 \eta^{\mu\nu} - p_1^\nu p_2^\mu) \Delta_3 \Big\} \frac{\Lambda^{D-4}-\mu^{D-4}}{D-4} \Bigg] + \cdots .
\end{aligned}
\tag{A.3}
$$

with

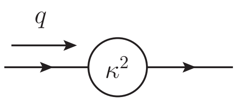
$$
\begin{aligned}
\Delta_1 &= \frac{4\delta(d-2)(d-1)}{d(d+2)} \\
\Delta_2 &= \frac{4((124\delta + d(88 - 28\delta + d(8 - 25\delta + 2d(-11 + 2d + 3\delta)))) - 96)}{d\,(d^2-4)} \\
\Delta_3 &= \frac{2\delta(d(24 + (23 - 6d)d) - 76) - 8(d+2)(d-4)(d(d-3)+1)}{d\,(d^2-4)}
\end{aligned}
$$

$$\mathcal{L}_s^{\text{c.t.}} = \frac{i}{(4\pi)^{1+\delta/2}\Gamma\left(\frac{\delta}{2}+2\right)}\left[\frac{2(8+5\delta)}{(\delta+2)^2}(\mathcal{D}_\mu\phi)^\dagger\mathcal{D}^\mu\phi\frac{\Lambda^{\delta+2}-\mu^{\delta+2}}{M_{(D)}^{\delta+2}} - \right.$$
$$\left. - \left\{(\mathcal{D}^2\phi)^\dagger\mathcal{D}^2\phi - \tfrac{1}{3}ig(\mathcal{D}_\mu\phi)^\dagger F^{\mu\nu}\mathcal{D}_\nu\phi + \tfrac{1}{6}g^2\phi^\dagger F^{\mu\nu}F_{\mu\nu}\phi\right\}\frac{\Lambda^\delta-\mu^\delta}{M_{(D)}^{\delta+2}}\right]. \quad \text{(A.4)}$$

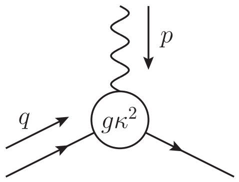
$$\delta_\phi = \frac{1}{(4\pi)^{D/2-1}\Gamma\left(\frac{D}{2}\right)M_{(D)}^{D-2}}\left(\frac{2(d-2)(d^2(D-3)+\delta(2-d))}{d(D-2)}\frac{\Lambda^{D-2}-\mu^{D-2}}{D-2}\right.$$
$$\left. - \frac{4d((2d(3-\delta)+11\delta)-20)-40\delta+64}{d(d-2)}m_\phi^2\frac{\Lambda^{D-4}-\mu^{D-4}}{D-4}\right) \quad \text{(A.5)}$$

$$\delta_{m_\phi^2} = \frac{1}{(4\pi)^{D/2-1}\Gamma\left(\frac{D}{2}\right)M_{(D)}^{D-2}}\left(\left(\frac{4(2-3d)\delta}{d(D-2)}+8d\right)\frac{\Lambda^{D-2}-\mu^{D-2}}{D-2}\right.$$
$$\left. - \frac{4(d-1)(\delta(d-10)-4(d-4))}{d(d-2)}m_\phi^2\frac{\Lambda^{D-4}-\mu^{D-4}}{D-4}\right) \quad \text{(A.6)}$$

A.2. Fermions



$$= \frac{i}{(4\pi)^{D/2-1}\Gamma\left(\frac{D}{2}\right)M_{(D)}^{D-2}}\left[m_\psi\left(\frac{1}{2(D-2)}(12-5D\right.\right.$$
$$+ d(d(4D-5)+5\delta-14))\frac{\Lambda^{D-2}-\mu^{D-2}}{D-2}$$
$$\left. + \frac{(5\delta+d(-5+d+3\delta))-12}{2(d-2)}m_\psi^2\frac{\Lambda^{D-4}-\mu^{D-4}}{D-4}\right)$$
$$- \not{q}\left(\frac{(d-1)(d((d(-7+4D)+3\delta)-16)-4\delta+8)}{2d(D-2)}\frac{\Lambda^{D-2}-\mu^{D-2}}{D-2}\right.$$
$$\left. + \frac{(10\delta+d(18-5\delta+d(d+3\delta-11)))-24}{2d(d-2)}m_\psi^2\frac{\Lambda^{D-4}-\mu^{D-4}}{D-4}\right)$$
$$+ \not{q}\not{q}m_\psi\frac{(d-1)\left((3d^2+(4+d)\delta)-16\right)}{2d(d-2)}\frac{\Lambda^{D-4}-\mu^{D-4}}{D-4}$$
$$\left. - \not{q}\not{q}\not{q}\frac{(d-1)(d((3d+\delta)-6)-6\delta+8)}{2d(d+2)}\frac{\Lambda^{D-4}-\mu^{D-4}}{D-4}\right] + \cdots \quad \text{(A.7)}$$



$$
\begin{aligned}
&= \frac{i\,gt^a}{(4\pi)^{D/2-1}\Gamma\left(\frac{D}{2}\right)M_{(D)}^{D-2}}\Bigg[-\gamma^\mu\Bigg(\frac{d-1}{2d(D-2)}\big(8-4\delta \\
&\quad + d(d(4D-7)+3\delta-16)\big)\frac{\Lambda^{D-2}-\mu^{D-2}}{D-2} \\
&\quad + \frac{(10\delta+d(18-5\delta+d(d+3\delta-11)))-24}{2d(d-2)}m_\psi^2\frac{\Lambda^{D-4}-\mu^{D-4}}{D-4}\Bigg) \\
&\quad + \{2q^\mu+p^\mu\}m_\psi\frac{(d-1)\left((3d^2+(4+d)\delta)-16\right)}{2d(d-2)}\frac{\Lambda^{D-4}-\mu^{D-4}}{D-4} \\
&\quad + \{\not{p}\gamma^\mu-\tfrac{1}{2}p^\mu\}m_\psi\frac{d(3\delta(d-1)+d(d+11))+36(\delta-4)}{2d(d-2)}\frac{\Lambda^{D-4}-\mu^{D-4}}{D-4} \\
&\quad - \Big\{2(\not{q}+\not{p})q^\mu+\not{q}\not{p}\gamma^\mu+\gamma^\mu(q^2+p^2)\Big\}\frac{1}{2d(d-2)}\big(134\delta-200 \\
&\quad + d\left(86-3d+d^2+3(d-7)\delta\right)\big)\frac{\Lambda^{D-4}-\mu^{D-4}}{D-4} \\
&\quad + \Big\{2(\not{q}+\not{p})q^\mu+(\not{q}+\not{p})p^\mu+\gamma^\mu q^2\Big\}\frac{1}{d\,(d^2-4)}\big(d(d(d((5d+7\delta)-17)-19\delta+32)+24\delta+60) \\
&\quad + 4(49\delta-72)\big)\frac{\Lambda^{D-4}-\mu^{D-4}}{D-4} \\
&\quad + \Big\{\not{q}(2q^\mu+p^\mu)+\gamma^\mu(q+p)^2\Big\}\frac{1}{d\,(d^2-4)}\big(d(d(d((5d+7\delta)-17)-19\delta+32)+24\delta+60) \\
&\quad + 4(49\delta-72)\big)\frac{\Lambda^{D-4}-\mu^{D-4}}{D-4} \\
&\quad + \Big\{\not{q}(2q^\mu+p^\mu)+\not{p}(q^\mu+p^\mu)+\gamma^\mu q\cdot(q+p)\Big\}\frac{1}{d\,(d^2-4)}\big(368-d(d(d((11D+2\delta)-41) \\
&\quad -35\delta+40)+4(3\delta+29))-252\delta\big)\frac{\Lambda^{D-4}-\mu^{D-4}}{D-4}\Bigg] + \cdots \qquad \text{(A.8)}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}_f^{\text{c.t.}} = \frac{1}{(4\pi)^{D/2-1}\Gamma\left(\frac{D}{2}\right)M_{(D)}^{D-2}}\overline{\psi}\Bigg[i\Bigg(&\frac{d-1}{2d(D-2)}(8-4\delta \\
&+d(d(4D-7)+3\delta-16))\frac{\Lambda^{D-2}-\mu^{D-2}}{D-2} \\
&+\frac{(10\delta+d(18-5\delta+d(d+3\delta-11)))-24}{2d(d-2)}m_\psi^2\frac{\Lambda^{D-4}-\mu^{D-4}}{D-4}\Bigg)\not{\mathcal{D}} \\
&-\Bigg(\frac{1}{2(D-2)}(12-5D \\
&+d(d(4D-5)+5\delta-14))\frac{\Lambda^{D-2}-\mu^{D-2}}{D-2} \\
&+\frac{(5\delta+d(-5+d+3\delta))-12}{2(d-2)}m_\psi^2\frac{\Lambda^{D-4}-\mu^{D-4}}{D-4}\Bigg)m_\psi \\
&+i\Bigg\{\frac{(1-d)\left((3d^2+(4+d)\delta)-16\right)}{2d(d-2)}m_\psi i\mathcal{D}^2 \\
&-\frac{d(3\delta(d-1)+d(d+11))+36(\delta-4)}{2d(d-2)}m_\psi gF_{\mu\nu}\gamma^{\mu\nu} \\
&+\frac{134\delta+d\left(86-3d+d^2+3(d-7)\delta\right)-200}{2d(d-2)}\not{\mathcal{D}}\not{\mathcal{D}}\not{\mathcal{D}} \\
&-\frac{d(d(d((5d+7\delta)-17)-19\delta+32)+24\delta+60)+4(49\delta-72)}{d\left(d^2-4\right)}\left(\not{\mathcal{D}}\mathcal{D}^2+\mathcal{D}^2\not{\mathcal{D}}\right) \\
&-\frac{1}{d\left(d^2-4\right)}(368-d(d(d((11D+2\delta)-41)-35\delta+40)+4(3\delta+29)) \\
&-252\delta)\mathcal{D}_\mu\not{\mathcal{D}}\mathcal{D}^\mu\Bigg\}\frac{\Lambda^{D-4}-\mu^{D-4}}{D-4}\Bigg]\psi
\end{aligned}
\tag{A.9}
$$

$$
\begin{aligned}
\delta_\psi = \frac{1}{(4\pi)^{D/2-1}\Gamma\left(\frac{D}{2}\right)M_{(D)}^{D-2}}\Bigg(&\frac{d-1}{2d(D-2)}(8-4\delta+d(d(4D-7) \\
&+3\delta-16))\frac{\Lambda^{D-2}-\mu^{D-2}}{D-2} \\
&+\frac{(10\delta+d(18-5\delta+d(d+3\delta-11)))-24}{2d(d-2)}m_\psi^2\frac{\Lambda^{D-4}-\mu^{D-4}}{D-4}\Bigg)
\end{aligned}
\tag{A.10}
$$

$$
\begin{aligned}
\delta_{m_\psi} = \frac{1}{(4\pi)^{D/2-1}\Gamma\left(\frac{D}{2}\right)M_{(D)}^{D-2}}\Bigg(&\frac{d((d(3D-5)+\delta)-6)-2\delta+4}{d(D-2)}\frac{\Lambda^{D-2}-\mu^{D-2}}{D-2} \\
&+\frac{(d-1)(3(d-4)+5\delta)}{d(d-2)}m_\psi^2\frac{\Lambda^{D-4}-\mu^{D-4}}{D-4}\Bigg)
\end{aligned}
\tag{A.11}
$$

A.3. Gauge Fields

The full results for the gauge fields can be found in chapter 5.

B. Example Form Script

In section 4.5, we illustrated the implementation of our computation in Form [46] by the example of the gravitational corrections to the scalar two-point function. Here, we present the script without inserted comments for those readers who want to test the example themselves and have a digital version of this thesis at hand.

```
*****************************************************
* Gravitational corrections to the scalar propagator *
*****************************************************

*gravitational coupling expansion parameter
symbol ka(:2);

*scalar mass, brane dimension & additional symbols for internal use
symbol MassS,D,[D+2],[D+4],[De-2],[de-2],de,De;

*fraction of the external momentum on the graviton line
symbol x;

*square root of metric determinat, inverse metric, graviton
tensors sqrtg,gI(s),h(s),H;

*momenta (external, loop & additional tensors for internal use)
tensors p,q,K,p1,p2,[-q],[-K],[K+p2],[-K-p2];

*function for symmetrisation
function Sym,P;

*indices (fixed)
dimension 0;
autodeclare index a;

*indices (summable)
dimension D;
indices mu,nu,a,b,c,d;
autodeclare index i;

*vector versions of the momenta
vectors qV,p1V,p2V;

Local Lphi2=sqrtg*(gI(mu,nu)*(-i_)*p(mu)*(-i_)*q(nu) -MassS^2);
```

```
repeat;
id,once,sqrtg=1+1/2*ka*h(i1,i1)
               +1/8*ka*ka*(h(i1,i1)*h(i2,i2)-2*h(i1,i2)*h(i1,i2));
id,once,gI(mu?,nu?)=d_(mu,nu)-ka*h(mu,nu)+ka*ka*h(mu,i1)*h(nu,i1);
sum i1,i2;
endrepeat;

.sort

id h(i1?,i2?)*h(i3?,i4?)=H(a1,a2,a3,a4)
                           *(d_(a1,i1)*d_(a2,i2)*d_(a3,i3)*d_(a4,i4)
                            +d_(a1,i3)*d_(a2,i4)*d_(a3,i1)*d_(a4,i2)
                            +d_(a2,i1)*d_(a1,i2)*d_(a3,i3)*d_(a4,i4)
                            +d_(a2,i3)*d_(a1,i4)*d_(a3,i1)*d_(a4,i2)
                            +d_(a1,i1)*d_(a2,i2)*d_(a4,i3)*d_(a3,i4)
                            +d_(a1,i3)*d_(a2,i4)*d_(a4,i1)*d_(a3,i2)
                            +d_(a2,i1)*d_(a1,i2)*d_(a4,i3)*d_(a3,i4)
                            +d_(a2,i3)*d_(a1,i4)*d_(a4,i1)*d_(a3,i2))/8;

id h(i1?,i2?)=h(a1,a2)*(d_(a1,i1)*d_(a2,i2)+d_(a2,i1)*d_(a1,i2))/2;

id ka=1;

bracket h,H;

.sort

functions fun1gr2scalar,fun2gr2scalar;

Local V1gr2scalar= i_*Lphi2[h(a1,a2)];
Local V2gr2scalar= i_*2*Lphi2[H(a1,a2,a3,a4)];

id p?(i1?!a1,a2,a3,a4)=p(i);
id d_(i1?!a1,a2,a3,a4,i2?)=d_(i,i2);

.sort

*propagator correction with 2 graviton-scalar (1+2) vertices (bubble)
Local PC1=-fun1gr2scalar(q,[-K-p2],a,b)*P(a,b,c,d)
                        *fun1gr2scalar([K+p2],[-q],c,d);

*propagator correction with 1 graviton-scalar (2+2) vertex (Seagull)
Local PC2=i_/2*fun2gr2scalar(q,[-q],a,b,c,d)*P(a,b,c,d);

*defining the vertex functions
repeat;
id,once,fun1gr2scalar(q?,p?,a1?,a2?)=V1gr2scalar;
sum i;
```

```
endrepeat;

repeat;
id,once,fun2gr2scalar(q?,p?,a1?,a2?,a3?,a4?)=V2gr2scalar;
sum i;
endrepeat;

*numerator of the graviton propagator
id P(a?,b?,c?,d?)=1/2*(d_(a,c)*d_(b,d)+d_(a,d)*d_(b,c))
                  -d_(a,b)*d_(c,d)/[De-2];

id [-q](i?)=-q(i);

id [-K-p2](i?)=-K(i)-p2(i);
id [K+p2](i?)=K(i)+p2(i);

bracket K;

.sort

id only K(mu?!a1,a2)*K(nu?!a1,a2)=K(a1)*K(a2)*Sym(a1,mu,a2,nu);
id only K(mu?!a1,a2)=K(a1)*d_(a1,mu);

id Sym(a1?,i1?,a2?,i2?)=(d_(a1,i1)*d_(a2,i2)+d_(a2,i1)*d_(a1,i2))/2;

bracket K;

.sort

L leadPC1=PC1[K(a1)*K(a2)]*Sym(a1,a2);
L leadPC2=PC2[1];

L subleadPC=PC1[1]*(De-2)/[de-2]
         -2*PC1[K(a1)]*(Sym(a1,p1V)+Sym(a1,p2V)*(De-2)/[de-2])
         +PC1[K(a1)*K(a2)]*(4*(Sym(a1,a2,p1V,p1V)*D/De
                              +Sym(a1,a2,p1V,p2V)
                              +Sym(a1,a2,p2V,p2V)*(De-2)/[de-2])
                           -(p1V.p1V
                             +(p2V.p2V-MassS^2)*(De-2)/[de-2])
                                                       *Sym(a1,a2));

id Sym(a1?,a2?,i1?,i2?)=(d_(a1,a2)*Sym(i1,i2)+d_(a1,i2)*Sym(i1,a2)
                         +d_(a1,i1)*Sym(i2,a2))/[D+2];
id Sym(a1?,a2?)=d_(a1,a2)/D;

.sort

index a1=D,a2=D,a3=D,a4=D;
```

```
*express momenta by a vector instead of a tensor
tovector q  qV;
tovector p1 p1V;
tovector p2 p2V;

.sort
if (expression(leadPC1,subleadPC));
id p1V=-qV*x;
id p2V=qV*(1-x);
endif;

.sort

L leadPC=leadPC1+leadPC2;

*remove the * star to fix the momentum parametrisation
*id x=0;

id D^x?=de^x;
id [D+2]^x?=(de+2)^x;
id [D+4]^x?=(de+4)^x;
id [de-2]^x?=(de-2)^x;
id [De-2]^x?=(De-2)^x;

bracket qV,MassS;

format mathematica;

print leadPC,subleadPC;

.end
```

Bibliography

[1] G. 't Hooft and M. J. G. Veltman, "One loop divergencies in the theory of gravitation," *Annales Poincare Phys. Theor.* **A20** (1974) 69–94. `http://www.numdam.org/numdam-bin/recherche?h=nc&id=AIHPA_1974__20_1_69_0`.

[2] S. Deser and P. van Nieuwenhuizen, "One Loop Divergences of Quantized Einstein-Maxwell Fields," *Phys. Rev.* **D10** (1974) 401.

[3] S. Deser and P. van Nieuwenhuizen, "Nonrenormalizability of the Quantized Dirac-Einstein System," *Phys. Rev.* **D10** (1974) 411.

[4] S. Deser, H.-S. Tsao, and P. van Nieuwenhuizen, "Nonrenormalizability of Einstein Yang-Mills Interactions at the One Loop Level," *Phys. Lett.* **B50** (1974) 491.

[5] S. Deser, H.-S. Tsao, and P. van Nieuwenhuizen, "One Loop Divergences of the Einstein Yang-Mills System," *Phys. Rev.* **D10** (1974) 3337.

[6] S. Deser and P. van Nieuwenhuizen, "Nonrenormalizability of the Quantized Einstein-Maxwell System," *Phys. Rev. Lett.* **32** (1974) 245–247.

[7] J. F. Donoghue, "General relativity as an effective field theory: The leading quantum corrections," *Phys. Rev.* **D50** (1994) 3874–3888, `arXiv:gr-qc/9405057`.

[8] J. F. Donoghue, "The Ideas of gravitational effective field theory," `arXiv:hep-th/9409143`.

[9] C. P. Burgess, "Introduction to effective field theory," *Ann. Rev. Nucl. Part. Sci.* **57** (2007) 329–362, `arXiv:hep-th/0701053`.

[10] C. P. Burgess, "Quantum gravity in everyday life: General relativity as an effective field theory," *Living Reviews in Relativity* **7** (2004) 5. `http://www.livingreviews.org/lrr-2004-5`.

[11] S. P. Robinson and F. Wilczek, "Gravitational correction to running of gauge couplings," *Phys. Rev. Lett.* **96** (2006) 231601, `arXiv:hep-th/0509050`.

[12] A. R. Pietrykowski, "Gauge dependence of gravitational correction to running of gauge couplings," *Phys. Rev. Lett.* **98** (2007) 061801, `arXiv:hep-th/0606208`.

[13] D. J. Toms, "Quantum gravity and charge renormalization," *Phys. Rev.* **D76** (2007) 045015, `arXiv:0708.2990 [hep-th]`.

[14] D. J. Toms, "Cosmological constant and quantum gravitational corrections to the running fine structure constant," *Phys. Rev. Lett.* **101** (2008) 131301, `arXiv:0809.3897 [hep-th]`.

[15] D. Ebert, J. Plefka, and A. Rodigast, "Absence of gravitational contributions to the running Yang-Mills coupling," *Phys. Lett.* **B660** (2008) 579–582, `arXiv:0710.1002 [hep-th]`.

[16] A. Rodigast, "One-loop divergences of the yang-mills theory coupled to gravitation," Master's thesis, Humboldt-Universität zu Berlin, 2007. `http://qft.physik.hu-berlin.de/`.

[17] D. Ebert, J. Plefka, and A. Rodigast, "Gravitational Contributions to the Running Yang-Mills Coupling in Large Extra-Dimensional Brane Worlds," *JHEP* **02** (2009) 028, `arXiv:0809.0624 [hep-ph]`.

[18] Y. Tang and Y.-L. Wu, "Gravitational Contributions to the Running of Gauge Couplings," *Commun.Theor.Phys.* **54** (2010) 1040–1044, `arXiv:0807.0331 [hep-ph]`.

[19] Y. Tang and Y.-L. Wu, "Quantum Gravitational Contributions to Gauge Field Theories," `arXiv:1012.0626 [hep-ph]`.

[20] Y. Tang and Y.-L. Wu, "Gravitational Contributions to Gauge Green's Functions and Asymptotic Free Power-Law Running of Gauge Coupling," *JHEP* **1111** (2011) 073, `arXiv:1109.4001 [hep-ph]`.

[21] J.-E. Daum, U. Harst, and M. Reuter, "Running Gauge Coupling in Asymptotically Safe Quantum Gravity," *JHEP* **1001** (2010) 084, `arXiv:0910.4938 [hep-th]`.

[22] D. J. Toms, "Quantum gravitational contributions to quantum electrodynamics," *Nature* **468** (2010) 56–59, `arXiv:1010.0793 [hep-th]`.

[23] H.-J. He, X.-F. Wang, and Z.-Z. Xianyu, "Gauge-Invariant Quantum Gravity Corrections to Gauge Couplings via Vilkovisky-DeWitt Method and Gravity Assisted Gauge Unification," *Phys.Rev.* **D83** (2011) 125014, `arXiv:1008.1839 [hep-th]`.

[24] N. Nielsen, "The Einstein-Maxwell system, Ward identities, and the Vilkovisky construction," *Annals Phys.* **327** (2012) 861–892, `arXiv:1109.2699 [hep-th]`.

[25] S. Folkerts, D. F. Litim, and J. M. Pawlowski, "Asymptotic freedom of Yang-Mills theory with gravity," *Phys.Lett.* **B709** (2012) 234–241, `arXiv:1101.5552 [hep-th]`.

[26] J. Felipe, L. Brito, M. Sampaio, and M. Nemes, "Quantum gravitational contributions to the beta function of quantum electrodynamics," *Phys.Lett.* **B700** (2011) 86–89, `arXiv:1103.5824 [hep-th]`.

[27] M. M. Anber, J. F. Donoghue, and M. El-Houssieny, "Running couplings and operator mixing in the gravitational corrections to coupling constants," *Phys.Rev.* **D83** (2011) 124003, `arXiv:1011.3229 [hep-th]`.

[28] J. Ellis and N. E. Mavromatos, "On the Interpretation of Gravitational Corrections to Gauge Couplings," `arXiv:1012.4353 [hep-th]`.

[29] J. Ellis and N. E. Mavromatos, "Does gravity correct gauge couplings?," *Nature* **479** (2011) E5–E6.

[30] D. J. Toms, "Quadratic divergences and quantum gravitational contributions to gauge coupling constants," *Phys.Rev.* **D84** (2011) 084016.

[31] O. Zanusso, L. Zambelli, G. Vacca, and R. Percacci, “Gravitational corrections to Yukawa systems,” *Phys.Lett.* **B689** (2010) 90–94, arXiv:0904.0938 [hep-th].

[32] A. Rodigast and T. Schuster, “Gravitational Corrections to Yukawa and φ^4 Interactions,” *Phys. Rev. Lett.* **104** (2010) 081301, arXiv:0908.2422 [hep-th].

[33] I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, and G. R. Dvali, “New dimensions at a millimeter to a Fermi and superstrings at a TeV,” *Phys. Lett.* **B436** (1998) 257–263, arXiv:hep-ph/9804398.

[34] N. Arkani-Hamed, S. Dimopoulos, and G. R. Dvali, “The hierarchy problem and new dimensions at a millimeter,” *Phys. Lett.* **B429** (1998) 263–272, arXiv:hep-ph/9803315.

[35] I. Antoniadis, “A Possible new dimension at a few TeV,” *Phys. Lett.* **B246** (1990) 377–384.

[36] K. R. Dienes, E. Dudas, and T. Gherghetta, “Grand unification at intermediate mass scales through extra dimensions,” *Nucl. Phys.* **B537** (1999) 47–108, arXiv:hep-ph/9806292.

[37] B. Grinstein, D. O’Connell, and M. B. Wise, “The Lee-Wick standard model,” *Phys. Rev.* **D77** (2008) 025012, arXiv:0704.1845 [hep-ph].

[38] T. D. Lee and G. C. Wick, “Negative Metric and the Unitarity of the S Matrix,” *Nucl. Phys.* **B9** (1969) 209–243.

[39] T. D. Lee and G. C. Wick, “Finite Theory of Quantum Electrodynamics,” *Phys. Rev.* **D2** (1970) 1033–1048.

[40] W. Pauli and F. Villars, “On the Invariant regularization in relativistic quantum theory,” *Rev. Mod. Phys.* **21** (1949) 434–444.

[41] B. Grinstein, D. O’Connell, and M. B. Wise, “Massive Vector Scattering in Lee-Wick Gauge Theory,” *Phys. Rev.* **D77** (2008) 065010, arXiv:0710.5528 [hep-ph].

[42] R. E. Cutkosky, P. V. Landshoff, D. I. Olive, and J. C. Polkinghorne, “A non-analytic S matrix,” *Nucl. Phys.* **B12** (1969) 281–300.

[43] A. van Tonder, “Unitarity, Lorentz invariance and causality in Lee-Wick theories: An asymptotically safe completion of QED,” arXiv:0810.1928 [hep-th].

[44] F. Wu and M. Zhong, “The Lee-Wick Fields out of Gravity,” *Phys. Lett.* **B659** (2008) 694–697, arXiv:0705.3287 [hep-ph].

[45] F. Wu and M. Zhong, “The TeV Scale Lee-Wick Fields out of Large Extra Dimensional Gravity,” *Phys. Rev.* **D78** (2008) 085010, arXiv:0807.0132 [hep-ph].

[46] J. A. M. Vermaseren, “New features of FORM,” arXiv:math-ph/0010025. http://www.nikhef.nl/~form/maindir/maindir.html.

[47] L. Randall and R. Sundrum, “A large mass hierarchy from a small extra dimension,” *Phys. Rev. Lett.* **83** (1999) 3370–3373, arXiv:hep-ph/9905221.

[48] G. Dvali, G. Gabadadze, and M. Porrati, "4-D gravity on a brane in 5-D Minkowski space," *Phys.Lett.* **B485** (2000) 208–214, `arXiv:hep-th/0005016 [hep-th]`.

[49] V. N. Popov and L. D. Faddeev, "Perturbation theory for gauge-invariant fields,". In *Lai, C. H. (ed.): Gauge Theory Of Weak and Electromagnetic Interactions*, 213-233 and Kiev Inst. Theor. Phys. Acad. Sci. - ITF-67-036 (67,REC.APR.68) 28 P. (803533) (SEE BOOK INDEX).

[50] R. Sundrum, "Effective field theory for a three-brane universe," *Phys. Rev.* **D59** (1999) 085009, `arXiv:hep-ph/9805471`.

[51] P. van Nieuwenhuizen, "Classical gauge fixing in quantum field theory," *Phys.Rev.* **D24** (1981) 3315.

[52] R. Woodard, "The vierbein is irrelevant in perturbation theory," *Phys.Lett.* **B148** (1984) 440.

[53] A. Rodigast and T. Schuster, "No Lee-Wick Fields out of Gravity," *Phys. Rev.* **D79** (2009) 125017, `arXiv:0903.3851 [hep-ph]`.

[54] S. Weinberg, *The Quantum Theory of Fields. Vol. 2: Modern applications*, ch. 18.2. Cambridge Univ. Pr., 1996.

[55] M. R. Niedermaier, "Gravitational Fixed Points from Perturbation Theory," *Phys.Rev.Lett.* **103** (2009) 101303.

[56] D. Benedetti and S. Speziale, "Perturbative quantum gravity with the Immirzi parameter," *JHEP* **1106** (2011) 107, `arXiv:1104.4028 [hep-th]`.

[57] R. Contino, L. Pilo, R. Rattazzi, and A. Strumia, "Graviton loops and brane observables," *JHEP* **06** (2001) 005, `arXiv:hep-ph/0103104`.

[58] A. Rodigast and T. Schuster, "Gravitational corrections to non-gauge interactions," *Nucl. Phys. Proc. Suppl.* **216** (2011) 263–264.

[59] H. Gies, "Introduction to the functional RG and applications to gauge theories," `arXiv:hep-ph/0611146 [hep-ph]`.

[60] M. Reuter and F. Saueressig, "Functional Renormalization Group Equations, Asymptotic Safety, and Quantum Einstein Gravity," `arXiv:0708.1317 [hep-th]`.

[61] E. Manrique, S. Rechenberger, and F. Saueressig, "Asymptotically Safe Lorentzian Gravity," *Phys.Rev.Lett.* **106** (2011) 251302, `arXiv:1102.5012 [hep-th]`.

[62] C. Wetterich, "Exact evolution equation for the effective potential," *Phys.Lett.* **B301** (1993) 90–94.

[63] L. Abbott, "The Background Field Method Beyond One Loop," *Nucl.Phys.* **B185** (1981) 189.

[64] R. Jackiw, "Gauge covariant conformal transformations," *Phys. Rev. Lett.* **41** (1978) 1635.

[65] D. Benedetti, P. F. Machado, and F. Saueressig, "Asymptotic safety in higher-derivative gravity," *Mod.Phys.Lett.* **A24** (2009) 2233–2241, `arXiv:0901.2984 [hep-th]`.

[66] C. D. Carone and R. F. Lebed, "A Higher-Derivative Lee-Wick Standard Model," *JHEP* **01** (2009) 043, `arXiv:0811.4150 [hep-ph]`.

List of Publications

- D. Ebert, J. Plefka, A. Rodigast, “Absence of gravitational contributions to the running Yang-Mills coupling”, *Phys. Lett. B* **660** (2008) 579, `arXiv:0710.1002 [hep-th]`.

- D. Ebert, J. Plefka, A. Rodigast, “Gravitational Contributions to the Running Yang-Mills Coupling in Large Extra-Dimensional Brane Worlds”, *J. High Energy Phys.* **0902** (2009) 028, `arXiv:0809.0624 [hep-ph]`.

- A. Rodigast, T. Schuster, “No Lee-Wick Fields out of Gravity”, *Phys. Rev. D* **79** (2009) 125017, `arXiv:0903.3851 [hep-ph]`.

- A. Rodigast, T. Schuster, “Gravitational Corrections to Yukawa and ϕ^4 Interactions”, *Phys. Rev. Lett.* **104** (2010) 081301, `arXiv:0908.2422 [hep-th]`.

- A. Rodigast, T. Schuster, “Gravitational corrections to non-gauge interactions,” *Nucl. Phys. Proc. Suppl.* **216**, 263-264 (2011).

Acknowledgement

First of all, I would like to express my gratitude to my supervisor Prof. Jan Plefka for his time, helpful advises, and encouragement. Furthermore, I would like to thank Prof. Dietmar Ebert for dedicating plenty of time to support this work and for being devotedly active in teaching and reminding students to keep in mind to regard the overall picture in physics and life.
I would like to devote special thanks to all former and present members of the Quantum Field Theory and String Theory group at the Humboldt University Berlin for the friendly and personal atmosphere, especially Theodor Schuster who was always open to discussions for the fruitful collaboration.
Last but no least, I thank my family, namely my parents Marlies and Rainer, my grandmother Edeltraut and my sister Sabine for their constant support throughout my life and helping me to keep up my studies.

Selbständigkeitserklärung

Ich erkläre, dass ich die vorliegende Arbeit selbständig und nur unter Verwendung der angegebenen Literatur und Hilfsmittel angefertigt habe.

Berlin, den 9. Januar 2012 Andreas Rodigast